FRAMEWORK FOR DIAGNOSTIC ASSESSMENT OF MATHEMATICS

Edited by
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NEMZETI TANKÖNYVKIADÓ

Framework for Diagnostic Assessment of Mathematics

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Developing Diagnostic Assessment
Projekt ID: TÁMOP 3.1.9-08/1-2009-0001

National Development Agency
www.ujszechenyiterv.gov.hu
06 40 638 638

HUNGARY'S RENEWAL

The project is supported by the European Union
and co-financed by the European Social Fund.

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ISBN 978-963-19-7217-7

Nemzeti Tankönyvkiadó Zrt.
a Sanoma company

www.ntk.hu • Customer service: info@ntk.hu • Telephone: 06-80-200-788

Responsible for publication: János Tamás Kiss chief executive officer
Storing number: 42684 • Technical director: Etelka Vasvári Babicsné
Responsible editor: Katalin Fried • Technical editor: Tamás Kiss
Size: 28,6 (A/5) sheets • First edition, 2011

Nessuna umana investigazione si può dimandare vera scienza,
se essa non passa per le matematiche dimostrazioni.

No human investigation can be called real science if it cannot be demonstrated mathematically.

Leonardo da Vinci

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Introduction

As suggested by the quote from Leonardo chosen as the motto for the present volume, mathematics plays a special role in the development of science. It has similarly special significance in formal education. It is the oldest of sciences, and even its early achievements continue to be part of the school curriculum today. It is one of the first fields of knowledge to be arranged into a school subject, and still tends to be the subject assigned the highest number of school periods. In the Hungarian education system mathematics is the only subject taught throughout the twelve grades of schooling. Children start preparing for formal mathematics education even before they start school and it remains a core subject in all science and engineering degree programs as well as in a substantial share of social science degree programs in higher education.

The study of mathematics has always been intertwined with the development of thinking and the acquisition of the ability of abstraction and logical reasoning. Mathematics also plays a role in solving everyday problems and the ability to use mathematical knowledge is an indispensable skill in several jobs. It is this special significance that has earned mathematics a permanent place among the assessment domains of the large-scale international comparative surveys, the results of which are taken into account when the development potential of participating countries is estimated. In Hungary, besides reading, the annual comprehensive educational assessment program also covers mathematics, and it has been naturally included together with reading and science in the project undertaking to develop a diagnostic assessment system.

Over the decades around the turn of the Millennium, research in education sciences and psychology has produced results that – if integrated and transferred into practice – may bring about a major turn in the improvement of the efficiency of education. The project providing the framework for the present volume occupies the intersection of three major research trends.

One of the key factors in the development of education systems is the availability of increasingly frequent, accurate and detailed feedback mechanisms at the different levels of decision making. In this respect, the most spectacular change in the past few decades was brought about by the development that the large-scale international surveys became regular events. The international comparative data enables us to identify the system-wide attributes of public education and the results of the consecutive assessment cycles provide feedback on the effects of any interventions. The methodological solutions of the international assessment programs have assisted the development of national assessment systems, and in several countries, including Hungary, an annual assessment program has been implemented providing primarily institutional level feedback. Through an analysis of their own survey data, institutions can improve their internal processes and activities, and as the results are made public, this may act as an incentive to seek ways of improvement. The experiences of countries where a system of this sort has been in place for a relatively long time show, however, that placing this sort of pressure on schools has the effect of improved efficiency only within certain limits, and too much pressure may lead to various distortions. Methods and tools directly assisting the work of teachers are indispensable for further improvement in performance. In this respect, the next stage of the development of evaluation can only be reached through the construction of systems suitable for providing frequent and detailed student-level feedback.

Traditional paper-based tests are not suitable for sufficiently frequent student assessment. For this reason, in the past teachers did not have access to measurement tools directly assisting learning through following student progress and signaling possible delays in good time. The second key factor we highlight is therefore the explosive advancement of information and communication technologies, which offer novel solutions in every area of life. The availability of these technologies in education allows the simple implementation of tasks that were previously impracticable, such as frequent educational assessments providing diagnostic feedback. Computers were put in the service of education effectively as soon as the first large electronic computers appeared; educational computer software was already available several decades ago. The use of information technology in education was, however, often motivated by the technology itself, i.e., the reasoning was that now that these tools were available, they might as well be used in education. Online diagnostic assessment reached this conclusion coming from the opposite direction, as an appropriate technology was sought for the implementation of an educational task of crucial importance. In our case, info-communication technology is a system component that has no substitute and that expands the range of possibilities for educational assessment.

The third factor, one which is closest to the concerns of this volume, is the cognitive revolution in psychology, which had an impact in several areas at the end of the last century and gave a new impetus to research efforts in connection with school learning and teaching. It has led to the emergence of new and more differentiated conceptions of knowledge, which have made it possible to define educational objectives more precisely and to develop scientifically based standards. This process also paved the way to a more detailed characterization of student development processes.

With the recognition of the crucial role of early childhood development, the focus of attention shifted to the initial stage of schooling, especially to the encouragement of language development and to the fostering of reasoning skills. Several studies have provided evidence that the acquisition of basic skills is indispensable for in-depth understanding of the subject matter taught at schools, which is in turn essential for students to be able to apply their knowledge to new contexts rather than just reproduce exactly what they have been taught. If the required foundations are not constructed, serious difficulties will arise at later stages of learning and the failures suffered during the first years of education will delimit students' attitudes towards education for the rest of their lives.

School mathematics plays an outstanding role in the development of cognitive abilities. In comparison with other subjects, it presupposes relatively little prior knowledge, thus its education can start at a very young age, in early childhood. Learning mathematics provides opportunities for students to recognize regularities, to weigh different options and to construct models. Very early on in mathematics education students can be encouraged to question what is believed to be true and to look for causes and proofs. Mathematics provides unique opportunities for understanding the significance of verification and proof. We have access to an enormous body of unstructured information and data. Mathematics can improve the skills needed for classifying data and information and for drawing the correct conclusions. There is a growing need for an ability to recognize and verify relationships, which is an issue that should be addressed in education. Science and technology advance at an enormous rate and factual knowledge may rapidly become out-of-date. Reasoning and problem solving skills, in contrast, never become obsolete and are needed in a growing number of areas in life. An important task of mathematics education right from the first grade of school is the development of reasoning and problem-solving skills, any deficiencies in which cannot be compensated for at later stages.

In line with the above trends, the Centre for Research in Learning and Instruction at the University of Szeged launched the project “Developing Diagnostic Assessments” in which the frameworks for diagnostic assessments in the domains of reading, mathematics and science have been developed. The current volume presents the results of our research in the domain of mathematics. Based on these results, assessment instruments, item banks of several hundred tasks covering the first six grades of school is constructed as a part of an online testing system. This system – the implementation of which is a lengthy process involving several hierarchically organized steps – will be suited to providing regular and frequent student level feedback on the various dimensions of changes in knowledge.

Diagnostic tests first of all give an indication of individual students’ state of development relative to various reference points. As in the case of system-wide surveys, the population average may act as a natural standard of comparison: The performance of a given student relative to his or her peers’ performance is an important piece of information. Online diagnostic tests, however, go even further; the system keeps a record of the students’ results allowing their progress and the evolution of their knowledge to be monitored over time.

The tools of measurement are based on content frameworks resting on scientific foundations, which are outlined in three volumes of parallel structure. The present volume discusses the frameworks for the assessment of mathematics while the two companion volumes are devoted to reading and science. The development work for the three domains proceeded in parallel and the same broad theoretical framework and conceptual system were used for the development of the detailed contents of their assessment. Besides having an identical structure, the three volumes also contain some identical sections in their introduction and in Chapter 4.

The work reported in this volume draws on the experiences of several decades’ research on educational assessment at the University of Szeged and on the achievements of the University of Szeged and Hungarian Academy of Sciences’ Research Group on the Development of Competencies, with special reference (a) to the results of studies related to the structure and organization of knowledge, educational evaluation, measurement theory, conceptual development, the development of reasoning skills, problem-solving and the assessment of school readiness, and (b) to the technologies developed for test item writing and test development. Constructing theoretical foundations for diagnostic assessments is, however, a complex task requiring extensive collaborative effort in the scientific community. Accordingly, the development of the frameworks has been a local and international co-operative enterprise involving researchers in the fields that are to be assessed. The opening chapter of each volume has been prepared with the contribution of a prominent specialist in the relevant field; thus our work rests upon scientific knowledge on the cutting edge of international research. The details of the frameworks have been developed by researchers and teachers and other professionals with practical experience in curriculum development and test construction.

The frameworks are based on a three-dimensional conception of knowledge in line with a tradition characterizing the entire history of organized education. The wish to educate the intellect, to cultivate thinking and general cognitive abilities is an age-old ambition. Modern education also sets several goals applying to the learners themselves as individuals. In order to attain these objectives we must first of all be guided by the achievements of scientific fields concerned with the human being and the developing child. This dimension can therefore draw on the results of developmental psychology, the psychology of learning and, more recently, on the achievements of cognitive neuroscience. With respect to the domain of mathematics, the core issue in this dimension is the development of mathematical thinking and skills.

Another set of objectives is related to the usability of the knowledge acquired at school. The dictum "Non scholae sed vitae discimus." is perhaps more topical today than every before, since our modern social environment is changing far too rapidly for public education to be able to keep pace with it. As revealed by previous research, the transfer of knowledge to novel contexts is not an automatic process; special teaching methods are called for in order to improve the skills of knowledge application. For this reason, it is essential that the question of the application of knowledge should appear as an independent dimension in the frameworks of diagnostic assessments. This constitutes a different system of goals, for which we must define what is expected of students that will enable them to apply their knowledge in different school contexts and in contexts outside of the school.

The third important issue is the question of which elements of the knowledge accumulated by the sciences and the arts should be selected as contents to be imparted at the school. Not only because the above objectives cannot be attained without content knowledge but also because it is an important goal of its own right that students should become familiar with the given domain of culture, the knowledge generated by mathematics and science and organized according to the internal values of a given scientific discipline. Mathematics is not only a tool for the development of reasoning and practical problem-solving skills but also an autonomous discipline of science with its own internal logic and factual content, which students should acquire adhering to the field's organizing principles and structure. Although the first grades of primary education focus on students' personal development and on the development of skills, neither efforts to improve cognitive abilities nor efforts to prepare children for practical problem solving can be successful in the absence of meaningful acquisition of scientific knowledge.

The above goals have been competing with each other over the past few decades with one or another coming into fashion or gaining dominance at different times at the expense of the others. For the purposes of this project, we assume that while education strives to achieve these objectives in an integrated way, they should be treated as distinct dimensions in diagnostic assessments. The surveys must be able to show if there is insufficient progress in one or another of these dimensions.

The first three chapters of this volume summarize the theoretical background and research evidence related to the three dimensions mentioned above. In Chapter 1, Terezinha Nunes and Benő Csapó provide an overview of psychological issues related to the development, fostering and assessment of mathematical thinking. This chapter discusses the natural process of the development of reasoning using numbers and quantities, which may be encouraged and enhanced by efficient mathematics instruction. In Chapter 2, Csaba Csíkos and Lieven Verschaffel summarize research results related to mathematical literacy and the application of mathematical knowledge. Chapter 3 by Julianna Szendrei and Mária Szendrei discusses the organization of mathematics as a scientific discipline, what aspects of this knowledge are appropriate for teaching, what aspects are generally taught at schools, and what kind of content the science of mathematics offers for the task of developing mathematical thinking and for practical applications. Each chapter relies on extensive research literature and the detailed bibliographies can assist further research efforts. In Chapter 4, Csaba Csíkos and Benő Csapó discuss theoretical issues and practical solutions in the development of frameworks and outline the basic principles guiding the construction of the detailed contents of diagnostic assessments. This chapter serves as a link between the theoretical chapters and the detailed content descriptions.

Chapter 5, which is the longest chapter making up half of the entire volume, contains the detailed frameworks of diagnostic assessment. The purpose of this chapter is to provide a basis for the development of measurement tools, the test items. The contents of assessment are grouped according to the three dimensions mentioned above. For the purposes of diagnostic assessment, the first six grades of schooling are considered to constitute a continuous education process. The results of the assessments therefore place students according to their current level of development along scales spanning all six grades. The content specifications of assessment questions could also essentially form a single continuous unit. However, in an effort to allow greater transparency and to follow the traditions of educational standards, this process has been divided into three stages, each of which covers approximately two years. For the three dimensions, therefore, a total of nine content blocks are described, each of which includes four main areas of mathematics.

In their present state, the frameworks detailed in this volume should be seen as the first step in a long-term development process. They specify what is reasonable to measure and what the major dimensions of assessment are, given the present state of our knowledge. As the domains covered develop at a very rapid rate, however, the latest findings of science should be incorporated from time to time. The content specifications can be constantly updated on the basis of our experiences of item bank development and an analysis of the data provided by the diagnostic assessment in the future. Our theoretical models can also be revised through an evaluation of the test items and an analysis of relationships emerging from the data. In a few years' time we will be in a position to look at the relationship between the various areas of early development and later performance allowing us to establish the predictive and diagnostic validity of test items, which can be a further important source of information for the revision of theoretical frameworks.

In the preparation of this volume *Csaba Csíkos* played a prominent role. In addition to co-authoring three of the chapters, he also led the research team developing the detailed description of the contents of the assessment. Besides the authors of the chapters, several colleagues have contributed to the completion of this volume, whose support is gratefully acknowledged here. Special thanks are also due to the team responsible for the management and organization of the project, *Katalin Molnár*, *Judit Kléner* and *Diána Túri*. The development and final presentation of the content of the volume have benefited greatly from the comments of the reviewers of earlier versions. We would like to take this opportunity to thank *József Kosztolányi* and *Ödön Vancsó* for their valuable criticism and suggestions.

Benő Csapó and *Mária Szendrei*

1

Developing and Assessing Mathematical Reasoning

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Introduction

Mathematics is one of the oldest scientific disciplines and offers valid content for school curricula. There are obvious possibilities for the application of its basics in everyday life, but a great majority of mathematical knowledge is taught in the hope that learning mathematics, besides improving reasoning and cultivating the mind in general, can provide students with systematic ways of approaching a variety of problems and with tools for analyzing and modeling situations and events in the physical, biological and social sciences. The power of mathematics as a tool for understanding the world was proclaimed by Galileo in unambiguous terms when he wrote that this great book of the universe, which stands continually open to our gaze, cannot be understood unless one first learns to comprehend the language and to read the alphabet in which it is composed: the language of mathematics (in Sobel, 1999).

In contrast to mathematics, scientific research into teaching and learning mathematics is a relatively young discipline; it is about a century old. The questions considered worth investigating and the research methods used to answer these questions changed over time but one question remains central in developmental psychology and education: Does learning mathematics improve reasoning or is mathematics learning only open to those who have attained an appropriate level of reasoning to begin with? Improving general cognitive abilities is especially important in a rapidly changing social environment; thus an answer to this question is urgent.

Modern developmental psychology has seen the rapprochement of two seemingly divergent theories that seek to explain cognitive development. On the one hand, Piaget and his colleagues analyzed the forms of reasoning that seem to characterize children's thinking as they grow up, focusing on the child's problem solving strategies (i.e., their actions and inferences) and justifications for these strategies (Inhelder & Piaget, 1958; Piaget & Inhelder, 1974, 1975, 1976). On the other hand, Vygostky paved the way for a deeper understanding of how cultural systems of signs (such as number systems, graphing and algebra) allow students to record their own thoughts externally, and then think and talk about these external signs, making them into objects and tools for thinking (Vygostky, 1978).

A simple example can illustrate this point. When anyone asks us for the time, we immediately look at our watches. In everyday life and in science, we think about time in ways that are influenced by the mathematical relations embodied in clocks and watches. We say "a day has 24 hours" because we measure time in hours; the ratio between 1 day and hours is 24:1; the ratio hours to minutes is 60:1, and the ratio minutes to seconds is 60:1. We represent time through this cultural tool – the watch – and the mathematical relations embodied in the watch, which allow us to describe the duration of a day. This cultural tool enables us to make fine distinctions between different times and also structures the way we think about time. Without it, we could not make an appointment with a friend, for example at 11 o'clock, and then say to the friend: "I'm sorry, I am 10 minutes late". Our perception of time is not that precise that we would be able to know exactly the time-point in the day that corresponds to 11 o'clock and to tell the difference between 11 and 11 : 10. This is the Vygotskian side of the story.

The Piagetian story comes into play when we think about what children need to understand in order to learn to read the watch and to compare different points in time. Numbers on the face of the watch have two meanings: they show the hours and the minutes. In order to read the minutes, children need to be able to relate 1 and 5, 2 and 10, 3 and 15 etc. and in order to find the interval, for example, between 1 : 35 and 2 : 15, they need to know that the hour has 60 minutes, and add the minutes up to 2 o'clock to the minutes after 2 o'clock. All this thinking has to be applied to the tool in order for children to learn to use it. We do not dwell on further examples here: it seems quite clear that learning to use a watch requires an understanding of the relations between minutes, hours and the numbers on the face of the watch. Research shows that this can still be challenging for 8-year-olds (Magina & Hoyles, 1997).

In this chapter we focus alternatively on the forms of reasoning that are necessary insights for learning mathematics and on the learning of conventional mathematical signs that enable and structure reasoning. The chapter is divided in three sections: whole numbers, rational numbers and solving problems in the mathematics classroom. In each of these sections we attempt to identify issues related to the psychological principles of learning mathematics and its cultural tools. The chapter is written with a focus on mathematics learning in primary school, and considers research with children mostly in the age range 5 to 12. There is no attempt to cover research about older students and no assumption that the issues raised here will suffice to understand further mathematics learning.

Those reasoning processes which are at the center of mathematics education are shaped by pre-school experiences and are influenced by outside school activities as well. Reasoning abilities developed in mathematics are applied to learning other school subjects while learning experiences in other areas may advance the development of mathematical reasoning. Well designed science education activities, for example, may stimulate those thinking abilities which are essential in mathematics too, first of all by providing experiential basis and practicing in the field. However, there are issues in cognitive development, where mathematics education plays a dominant role, such as reasoning with quantities and measures, using mathematical symbols etc. In this chapter we focus on these issues discussing in more details their critical position in further and broader mathematical development. At the same time, we acknowledge the importance of the role that mathematics education plays in promoting several further reasoning processes, but in this chapter we deal with them only in brief.

Mathematics Education and Cognitive Development

Whole Numbers

The aim of learning numbers in the initial years of primary school is to provide children with symbols for thinking and speaking about quantities. Later on in school students may be asked to explore the concept of number in a more abstract way and to analyze number sequences that are not representations of quantities, but throughout most of the primary school years numbers will be used to represent quantities and relations between them.

Quantities and numbers are not the same. Thompson (1993) suggested that “a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person’s height, and the amount by which one of you is taller than the other without having to know the actual values” (pp 165–166). You can also know, without using numbers, that if you are taller than your friend Rick, and Rick is taller than his friend Paula, you are taller than Paula. You are certain of this even if you have never met Paula. So we can think about relations between quantities without having a number to represent them. But when we can represent them with numbers, we can know more: If you know that you are 4 cm taller than Rick and that Rick is 2 cm taller than Paula, you know that the difference between yours and Paula’s height is 6 cm.

In the initial years in primary school, children learn about numbers as tools for thinking and speaking about quantities. The emphasis we place here on numbers as representations rests on the significance of number systems as tools for thinking. We can’t record quantities or communicate with others about them if we do not have a number system to represent quantities. A system for representing quantities allows us to make distinctions that we may not be able to make without the system. For example, we may not be able to tell just from looking the difference between 15 and 17 buttons, but we have no problem in doing so if we count them. Or we may not be able to tell whether a cupboard we want to buy, which we see in a shop, fits into a space in our house, but we will know if we measure the cupboard and the space where we want it to fit. Systems of representation of quantities allow us to make distinctions we cannot make perceptually and make comparisons between quantities across time and space. They enable and structure our thinking: we think with the numbers we use in measuring.

Thus, there are two crucial insights that children need to attain in order to understand whole numbers. First, they must realize that their knowledge of numbers and of quantities should be connected. Second, they must understand how the number system works.

Piaget, and subsequently many other researchers, explored several ideas that children should have about the connections between quantities and their numerical representation. Children should know, for example, that:

(1) if two quantities are equivalent, they should be represented by the same number
(2) if two quantities are represented by the same number, they are equivalent
(3) if some items are added to a set, the number that represents the set should change and should be a larger number
(4) if some items are taken away from the set, the number should change and be a smaller number
(5) if the same number of items is added to and then subtracted from the set, the quantity and the number of items in the set do not change (i.e. they should understand the inverse relation between addition and subtraction).

These insights into the relationship between number and quantity do not seem to be available to children younger than about four or five years (see Ginsburg, Klein, & Starkey, 1998, for a review of the first four points; see Nunes, Bryant, Hallett, Bell, & Evans, 2009, for a review regarding the last point). There is research that suggests that young children, even babies, can see that, if you add one item to a set of one, you should have two items, but there is no evidence to indicate that babies know that there is a connection between quantities and numerical symbols. All testing in the studies with babies is perceptual, and thus they tell us nothing about knowing that a set represented by the number 1 should no longer be represented by 1 after you add items to it. The difference between perceptual judgments and the use of symbols is at the heart of understanding mathematical learning and reasoning.

These five insights regarding the connection between quantities and numbers are necessary (but not sufficient, as it will be argued later) for understanding whole numbers but they do not have the same level of difficulty.

The first four are considerably simpler than the last one, which we refer to as the inverse relation between addition and subtraction.

The difficulty of understanding the inverse relation between addition and subtraction results from the need to coordinate two operations, addition and subtraction, with each other and to understand how this coordination affects number; it does not result from the amount of information that the children need to consider in order to answer the question. Bryant (2007) demonstrated this in a study where children were asked to consider the same amount of information about sets; some problems involved the inverse relation between addition and subtraction whereas others did not. In the inverse problems, the same number of items were added to and subtracted from a single set. In the problems that did not involve inversion, the same number of items was added to one set and subtracted from an equivalent set. Some children were able to realize that the originally equivalent sets differed after items were added to one and subtracted from the other but nevertheless did not succeed in the inverse relation items, which involved operations on the same set.

If understanding the connection between quantities and their numerical representation really is important, there should be a relationship between children's insights into these connections and their learning of mathematics. Children who already realize how quantity and number are related when they start school should have an advantage in mathematics learning in comparison to those who did not attain these insights. Two studies carried out by different research teams (Nunes, Bryant, Evans, Bell, Gardner, Gardner, & Carraher, 2007; Stern, 2005) show that children's understanding of the inverse relation between addition and subtraction predicts their mathematical achievement at a later time, even after controlling for general cognitive factors such as intelligence and working memory.

Children's understanding of the inverse relation develops over time. Children are at first able to realize that there is an inverse relation between addition and subtraction if the problems are presented to them with the support of quantities (either visually available or imagined); later, they also seem to understand this when asked about numbers, with no reference to quantities. If asked what is 34 plus 29 minus 29, they know they do not need to compute the sums: they know that the answer is 34. They may be able to also know the answer to 34 + 29 – 28 without calculating, but this is a more difficult question.

In summary, in order to understand whole numbers, children must realize that there are specific connections between quantities and numbers. At about age 4 to 5, children understand that, when two sets are equivalent, if they count one set they know how many items are in the other without having to count. At about age 6, they are able to understand also the inverse relation between addition and subtraction, and know that the number does not change if the same number of items is added to and subtracted from a set. This insight is a strong predictor of later mathematics achievement.

The insights we described in the previous paragraphs are about the logical relations between quantities and numbers but this is not all one should consider when analyzing whole numbers. One must ask also: when numbers are represented using a base-ten system, what demands does the nature of this representation place on the learner's cognitive skills? The base-ten system places two demands on the learner's cognition: the learner must also have some insight into additive composition and into multiplicative relations.

Additive relations require thinking about part-whole relations. In order to understand what 25, for example, means, the learner should understand that the two parts, 20 and 5, together are exactly as much as the whole, 25. In more general terms, the learner must understand additive composition of numbers, which means that any number can be formed by the sum of two other numbers.

The multiplicative relations in the base ten system have to do with the way the number labels and the place value system work. When we write numbers, the place where the digit is indicates an implicit multiplication: if the digit is the last one on the right, it is multiplied by 1, the second to the left is multiplied by 10, and the third to the left is multiplied by 100 and so on.

Young children's understanding of these additive and multiplicative relations in the number system may be subtle and implicit so we need specific tasks to assess this knowledge. We have created tasks that seem to assess additive composition and early multiplicative reasoning, which can be used to predict children's mathematics achievement. Additive composition is assessed by our "Shop Task". We ask children to pretend to buy items in a shop; they are given coins of different values to buy the items. If they want to buy, for example, a toy car that costs 9 cents, and they have one 5-cent coin and six 1-cent coins, they need to combine the 5-cent coin with four 1-cent coins. Children who do not understand additive composition think that they do not have exact change to pay for the toy: they say that they have five and six cents but their money does not allow them to "make" 9 cents. About two thirds of children aged 6 years pass this question. This question is highly predictive of mathematics achievement later on in primary school (Nunes et al., 2007; Nunes, Bryant, Barros, & Sylva, 2011).

We assess young children's understanding of multiplicative relations by asking them to solve multiplication and division problems using objects. For example, we show them a row of four houses and invite them to imagine that inside each house live three rabbits. We then ask them how many rabbits live in these houses. Children who have some early understanding of multiplicative relations in action simply point three times to each house and "count the rabbits" as they point to the houses. Young children's ability to pass items such as this helps predict their mathematical achievement later on (Nunes et al., 2007; Nunes, Bryant, Barros, & Sylva, 2011).

In summary, children must attain two sorts of insights in order to understand whole numbers. They need to understand the connections between quantities and numbers, and they need to understand the principles implicit in the number system that we use to represent whole numbers, which is a base-ten system. Research indicates that children who attain these insights at the beginning of primary school show higher levels of mathematical achievement later on, when the children are 8, 11 and 14 years (Nunes, Bryant, Barros, & Sylva, 2011). So, early assessments of mathematics should include items that measure such insights in order to help teachers make decisions about what to teach to their children.

Rational Numbers

Rational numbers are needed to express parts of the whole. These quantities appear in measurement and quotient situations. In a measurement situation, for example, if you are measuring sugar with a cup and the quantity you have is less than a cup, you might describe it as a third of a cup – or, with numerical symbols, 1/3. In a quotient situation, for example, you might be sharing one chocolate among three children; each child receives the result of dividing 1 by 3, or 1/3. These two situations in which fractions are used have in common the fact that, in order to speak of fractions, a division in equal parts has to take place. Fractions, thus, are numbers that result from division, rather than from counting, as whole numbers do. (Here we always mean positive parts of positive wholes.)

A division has three terms

(1) a dividend, which is the quantity being divided
(2) a divisor, which is the number of parts into which the quantity is divided
(3) a quotient, which is the result of the division and the value represented by the fraction.

In order to understand fractions as representations of quantities, children need to understand the connections between these numbers and the quantities that they represent. Fractions differ from whole numbers in many ways: we consider three basic differences here that must be mastered by students if they are to understand these numbers.

(1) A term within a fraction is given meaning by its relation to the other term: thus by knowing only the numerator we can not tell the quantity represented by the fraction.
(2) The same fraction might represent different quantities when the fraction itself bears a relation to a whole. So ½ of 8 and ½ of 12 are not equivalent although they are expressed by the same fraction.
(3) Different fractions might represent the same quantity: ½ and 2/4 of the same pie represent the same quantity; this is treated in the mathematics classroom as the study of equivalent fractions.

Many students do not seem to understand at first that the numbers in a fraction represent relations between quantities (Vamvakoussi & Vosniadou, 2004); it takes some time for this understanding to develop, at least under the present conditions of instruction. We explore below some of the ways in which this aspect of understanding fractions has been investigated.

One relation that students must understand is that, the greater the dividend, the greater the quotient, if the divisor remains the same. In part-whole situations, the dividend is the whole, which is not explicit in the fractional numerical representation; when we say 1/3 cup, the quantity in a cup is what is being divided. It may be easy to understand that 1/3 of a small cup and 1/3 of a large cup will not be the same quantity. But perhaps it is not as easy for students to understand that the quantity represented by the symbol 1/3 may not always be the same because the quantity being divided may not be the same.

We know of no studies that included a question about whether the same fraction may represent different numbers (when expressing fractions of different wholes) but Hart, Brown, Kerslake, Kücherman, and Ruddock (1985) included in their large scale study of students' understanding of fractions a question that investigates students' understanding of the connection between fraction symbols and quantities. They told students that Mary spent ¼ of her pocket money and John spent ½ of his pocket money, and then asked: is it possible that Mary spent more money than John? If students understand that the size of the whole matters, they should say that it is indeed possible that Mary spent more money, although ½ is more than ¼ if the quantities come from the same whole. However, 42% of the 11–12 year olds and 34% of the 12–13 year olds said that it is not possible; they justified their answer by indicating that ½ is always more than ¼. So, it is not obvious to students in this age range that the same fraction might not represent equivalent quantities.

Understanding the equivalence of fractions – that is, that different fractions may represent the same quantity – is crucial for connecting quantities with fractional symbols and also for adding and subtracting fractions. Research suggests that fraction equivalence is not easy for many students (e.g. Behr, Wachsmuth, Post, & Lesh, 1984; Kerslake, 1986) and that this is not an all-or-nothing insight: students might attain this insight in one type of situation but not in another. We (Nunes, Bryant, Pretzlik, Bell, Evans, & Wade, 2007) investigated students' (age range 8 to 10 years) understanding of the equivalence of fractional quantities in the context of part-whole and quotient situations, both presented with the support of drawings. The problem in the part-whole situation was: Peter and Alan were given chocolate bars of the same size, which were too large to be eaten in one day. Peter broke his chocolate in 8 equal parts and ate 4; Alan broke his chocolate in 4 equal parts and ate 2. The students were asked whether the boys ate the same amount of chocolate. The rate of correct responses to this problem was 31%. The problem in the quotient situation was: a group of 4 girls is sharing equally one cake and a group of 8 boys is sharing equally two cakes which are identical to the girls' cake. The students were asked whether, after the division, each girl would eat the same amount of cake as each boy. The rate of correct responses in this situation was 73%. Thus, understanding the equivalence between fractional quantities seems to happen in different steps: quotient situations lead to significantly better performance.

The difference in students' performance between these two situations surprises many teachers but it is important to remember that problems that seem very similar to a mathematician can be perceived as completely different by students (Vergnaud, 1979). Developmental psychologists test whether children perceive different objects as instances of the same category by teaching them to name one object and asking them to name the second one, without any instruction. If the children generalize the name learned for the first object to the second, one can infer that they see both as instances of the same category.

This approach has been used in the analysis of fractions in two studies (Nunes, Campos & Bryant, 2011; Mamede, 2007). In these studies, two groups of students who had not yet received instruction about fractions in school were introduced to the use of fractional representation in an experiment. The students were randomly assigned to one condition of instruction: they either learned to use fraction symbols to represent part-whole relations or to represent quantities in quotient situations. Both groups of students progressed in the use of fractions symbols from pre- to post-test and made significantly more progress than a control group, but this progress was specific to the situation in which they received instruction. Students who learned to use fractions for part-whole relations could not use fractions to represent quotient situations, and vice-versa. So, children do not immediately see that they can use fractions to represent part-whole and quotient situations: they do not generalize the use of these symbols from one situation to the other. This finding should caution researchers about drawing general conclusions about students' knowledge of fractions if they have analyzed the students' performance in only one type of situation.

Finally, putting fractions in order of magnitude involves understanding the relationship between the divisor and the quotient in a division, or between the denominator and the quantity represented in a fraction: if the numerator is constant, the larger the denominator, the smaller the quantity represented. Children seem understand the inverse relation between the divisor and the quotient when they are focusing on quantities rather than symbols: a large proportion of 6- and 7- year olds understands, for example, that the more people sharing a cake (or a certain number of sweets), the less each one receives. However, this understanding does not translate immediately into knowledge of how fractions can be put in order of magnitude. Hart et al. (1985) asked students to place in order of magnitude the fractions ¼, ½, 1/100 and 1/3. This could be an easy item because the numerator is constant across fractions, but only about 2/3 of the students in the age range 11-13 ordered these fractions correctly.

In conclusion, rational numbers are required for representing quantities that arise in division situations, rather than as the result of counting. So, in order to understand the connection between the quantities represented by rational numbers and fraction symbols, students must understand the relations between the three quantities in a division situation. The same fraction may represent different quantities when they are fractions of different wholes. Two different fractions represent the same quantity when the relationship between the numerator and the denominator is the same, although the numerator and denominator of the two fractions are different. For fractions of the same numerator, the larger the denominator, the smaller the quantity represented. Finally, the generalization of the use of fraction symbols between part-whole and quotient situations is not obvious to students, and insights developed in quotient situations may not be transferred to part-whole situations, and vice-versa.

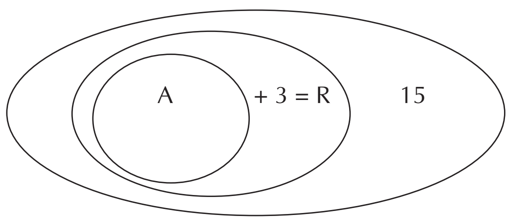
Solving Arithmetic Problems

Much attention in research about solving arithmetic problems has focused on learning to calculate with multi-digit numbers. This valuable research (e.g. Brown & VanLehn, 1982; Resnick, 1982) taught us much about the principled way in which children approach computations, even when they make errors. This research will not be discussed here because the levels of difficulty of calculation with the different types of multi-digit numbers is well documented: for example, it is known that calculation with regrouping (i.e. carrying or borrowing) is difficult; it is also known that subtracting, multiplying and dividing when there is a zero in the numbers is problematic, but zeros cause fewer problems in addition. So it is not difficult to choose a few computation problems that can offer a good assessment of computation skills. Unfortunately, the best way to teach students how to calculate remains controversial, as well as the very need to teach students the traditional written computation algorithms in the context of modern technological societies (see Nunes, 2008). In spite of this latter problem, this section focuses not on how to do sums but knowing when to do which sums.

In the first 6 to 8 years of primary school, students are taught mathematics that draws on two different types of relations between quantities: additive relations, based on part-whole relations between quantities, and multiplicative relations, based on correspondences (of different types) between quantities. The differences between these two types of relations are best understood if we consider an example. Figure 1.1 presents two problems and identifies the quantities and relations in each one.

Both problems describe a quantity, the total number of books that Rob and Anne have, and the relation between two quantities, Rob's and Anne's books. The relation between the quantities in Problem 1 is described in terms of a part-whole structure, as illustrated in the diagram. Part-whole relations are additive. The relation between the quantities in Problem 2 is described in terms of one-to-many correspondence, as illustrated in the diagram; these are multiplicative relations.

(1) Together Rob and Anne have 15 books (quantity). Rob has 3 more books than Anne (or Anne has 3 books fewer than Rob) (relation). How many books does each one have? (quantity)



(2) Together Rob and Anne have 15 books (quantity). Rob has twice the number of books that Anne has (or Anne has half the number of books that Rob has) (relation). How many books does each one have? (quantity)

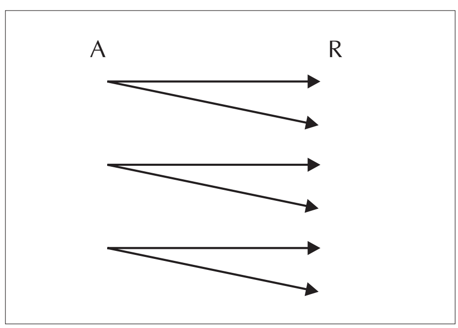


Figure 1.1 A schematic representation of relationships between quantities in additive and multiplicative situations

A major use of mathematics in problem solving involves the manipulation of numbers in order to arrive at conclusions about the problems without having to operate directly on the quantities: in other words, to model the world. To quote Thompson (1993): "Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships … . A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities" (p. 165). If students analyze the relationships between quantities in a way that represents the situation well, the mathematical model they build of the situation will be adequate, and the calculations that they implement will lead to correct predictions. If they analyze the relationships between quantities in a way that distorts the situation, the model they build of the situation will be inadequate, and the calculations that they implement will lead to incorrect predictions.

Some situations are immediately understood as additive or multiplicative, and young children, aged 5 and 6, can solve problems about these situations even before they know how to calculate. They use different actions in association with counting to solve these problems. Their actions reveal the way in which they establish relations between the quantities.

A great deal of research (e.g. Brown, 1981; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982; De Corte & Verschaffel, 1987; Kintsch & Greeno, 1985; Fayol, 1992; Ginsburg, 1977; Riley, Greeno, & Heller, 1983; Vergnaud, 1982) shows that pre-school children use the appropriate actions when solving problems that involve changes in quantities by addition or subtraction: to find the answers to these problems, they put together and count the items, or separate and count the relevant set. Very few pre-school children seem to know addition and subtraction facts; yet, when they are given the size of two parts, and asked to tell the size of the whole, their rate of correct responses is above 70%, if the numbers are small and they have no difficulty with counting. This is probably not surprising to most people.

However, most people seem surprised when they find out that such young children also show rather high rates of success in multiplication and division problems when they can use objects to help them answer the questions. Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) gave multiplicative reasoning problems to U.S. kindergarten children involving correspondences of 2 : 1, 3 : 1 and 4 : 1 between the sets (e.g. 2 sweets inside each cup;

how many sweets in 3 cups?). They observed 71% correct responses to these problems. Becker (1993) observed 81% correct responses to multiplicative reasoning problems among 5-year-olds in U.S. kindergartens, when the ratios between quantities were 2 : 1 and 3 : 1.

So, when objects are available for manipulation, young children distinguish easily between the actions they need to carry out to solve simple additive and multiplicative problems. However, the level of difficulty of different types of problems varies within both additive and multiplicative reasoning problems. Vergnaud (1982) argued that what makes many arithmetic problems difficult is not the numerical calculation that students need to carry out but the difficulty of understanding the relations involved in the problem situations. Vergnaud refers to this aspect of problem solving as the relational calculus, which he distinguishes from the numerical calculus – i.e. from the computation itself. In the subsequent sections, we discuss first the difficulties of relational thinking in the domain of additive reasoning and then in the domain of multiplicative reasoning.

Additive Reasoning Problems

Different researchers (e.g. Carpenter, Hiebert, & Moser, 1981; Riley, Greeno, & Heller, 1983; Vergnaud, 1982) proposed very similar classifications for the simplest forms of problems involving addition and subtraction. The basis of these classifications is the type of relational calculation involved. Three groups of problems are identified using this approach. In the first group problems, known as combine problems, were included problems about quantities which were combined (or separated) but not changed (e.g. Paul has 3 blue marbles and 6 purple marbles; how many marbles does he have altogether?). The second group, known as change problems, included problems that involved transformations from initial states resulting in final states (e.g. Paul had 6 marbles; he lost 4 in a game; how many does he have now?). The third group, known as comparison problems, included problems in which relational statements are involved (e.g. Mary has 6 marbles; Paul has 9 marbles; how many more marbles does Paul have than Mary?). The question "how many more marbles does Paul have than Mary" is a question about a relation rather than a quantity. It can be reformulated as "how many fewer marbles does Mary have than

Paul?" Relations have a converse ("how many more" has the converse "how many fewer?"); quantities do not.

The research carried out about these different types of problems showed that combine problems and change problems in which the initial state was known were the easiest problems. Children aged about 6 perform at ceiling level in these types of problems. However, the simplest comparison problems are still difficult for many 8 year olds whereas the most difficult ones, which involve thinking of the converse statement about the comparison, are still challenging for many students in the age 10–11 years. For example, Verschaffel (1994), working with a small sample of students in Belgium reported that if students were given the problem "Charles has 34 nuts. Charles has 15 nuts less than Anthony. How many nuts does Anthony have?", about 30% subtracted 15 from 34 and answered incorrectly. Lewis and Mayer (1987) reported that this error was still presented among U.S. college students, aged 18 years or older, but to a lesser degree (about 16%).

Combine problems always involve quantities and are relatively simple even when the number representing the quantities in the problem is increased. However, change problems involve transformations; combining transformations is more difficult than combining quantities and analyzing transformations is more difficult than separating quantities. For example, consider the two problems below, the first about combining a quantity and a transformation and the second about combining two transformations.

(1) Pierre had 6 marbles. He played one game and lost 4 marbles. How many marbles did he have after the game?

(2) Paul played two games of marbles. He won 6 in the first game and lost 4 in the second game. What happened, counting the two games together?

French children, who were between pre-school and their fourth year in school, consistently performed better on the first than on the second type of problem, even though the same arithmetic calculation (6 – 4) is required in both problems. By the second year in school, when the children are about 7-years-old, they achieved 80% correct responses in the first problem, and they only achieve a comparable level of success in the second problem two years later, when they were about 9 years. So, combining transformations is more difficult than combining a quantity and a transformation.

Three studies can be used to illustrate the difficulty of thinking about relations between quantities, two coming from a quantitative and one from a qualitative tradition.

This first example comes from the Chelsea Diagnostics Mathematics Tests (Hart, Brown, Kerslake, Kuchermann, & Ruddock, 1985), which includes three problems about relations. All three problems have distances as the problem content: distance is not a measure but a relation between two points. The simplest problem is "John is cycling 8 miles home from school. He stops at a sweet shop after 2 miles. How do you work out how much further John has to go?" The question was a multiple choice one, and included three possible answers involving addition and subtraction: 8 – 2, 2 – 8, and 2 + 6. The other four choices involved operations with either the multiplication or division signs. A total of 874 students participated in this study, whose ages were in the ranges 10–11, 11–12 and 12–13 years. The rate of correct responses did not show any increase between 10–11 and 12–13 years, and varied around 68% correct. The other two problems that were of a similar type showed a similar leveling of performance at about 70%. (One problem which had two correct answers showed a slightly higher percentage of correct responses, reaching 78% for the 11–12 year olds.)

The second example involves the use of positive and negative numbers and relations to solve a problem. Our own work (not published in this level of detail yet) illustrates this. The data came from a longitudinal study with two cohorts; both cohorts were tested when they were on average about 10 years 7 months (N = 7,981) and the first cohort was tested again in the same items when they were on average 12 years 8 months (N = 2,755).

The problem was about pinball games, in which the player's score depends on the number of balls placed in different parts of the board (see Figure 1.2). For each ball in the treasure zone, the player wins one point; for each ball in the skull zone, the player loses one point; no points are awarded for balls lost in the bottom. Obtaining the score for each game is a relatively simple question when all the points are positive: about 90% of the students correctly give the score for Game 3. The rate of correct responses goes down to 48% and 66%, respectively for the 10–11 and 12–13 age groups, when the player lost points. However, combining information about the end result with the information about these two games in order to indicate the player's score in the first game is a much more difficult task: only 29% of the students in the 10–11 year old group and 46% of the students in the 12–13 year old group were successful here. Because the numbers in the problems are small, it is not possible to explain the problem difficulty by the difficulty of the numerical calculus: the difficulty must be connected to the relational calculus. In the pinball game, positive and negative numbers have to be combined, and a relation between the score in two games and the final score must be used in order to infer what the score in the first game must have been.

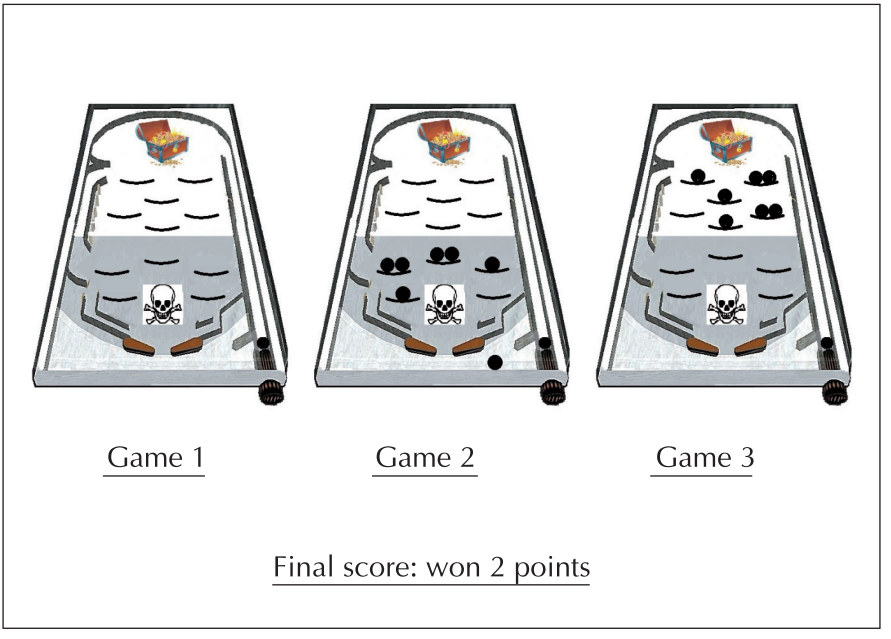


Figure 1.2 An example of a problem based on the pinball game

A few studies about directed numbers (positive and negative numbers) have been carried out in the past, which show that, when all the numbers have the same sign (i.e. are all positive or negative), students treat them as natural numbers, and then assign to them the sign that they had. But combining information from negative and positive numbers requires much more relational reasoning. Marthe (1979), for example, found that only 67% of the students in the age group 14–15 years were able to solve the problem "Mr. Dupont owes 684 francs to Mr. Henry. But Mr Henry also owes money to Mr. Dupont. Taking everything into account, it is Mr. Dupont who must give back 327 francs to Mr. Henry. What amount did Mr. Henry owe to Mr. Dupont?"

Finally, the third example is provided by Thompson's (1993) qualitative analysis of the difficulties that students encounter in distinguishing between relations and quantities in a study with 7- and 9-year olds. He analyzed students' reasoning in complex comparison problems which involved at least three quantities and three relations. His aim was to see how children interpreted complex relational problems and how their reasoning changed as they tackled more problems of the same type. To exemplify his problems, the first problem is presented here: „Tom, Fred, and Rhoda combined their apples for a fruit stand. Fred and Rhoda together had 97 more apples than Tom. Rhoda had 17 apples. Tom had 25 apples. How many apples did Fred have?" (p. 167). This problem includes three quantities (Tom's, Fred's and Rhoda's apples) and three relations (how many more Fred and Rhoda have than Tom; how many fewer Rhoda has than Tom; a combination of these two relations). He asked six children who had achieved different scores in a pre-test (three with higher and three with middle level scores) sampled from two grade levels, second (aged about 7) and fifth (aged about 10) to discuss six problems presented over four different days. The children were asked to think about the problems, represent them and discuss them.

On the first day the children went directly to trying out calculations and treated the relations as quantities: the statement "97 more apples than Tom" was interpreted as "97 apples". This led to the conclusion that Fred has 80 apples because Rhoda has 17. On the second day, working with problems about marbles won or lost during the games, the researcher taught the children to use representations for relations by writing, for example, "plus 12" to indicate that someone had won 12 marbles and "minus 1" to indicate that someone had lost 1 marble. The children were able to work with these representations with the researcher's support, but when they combined two statements, for example minus 8 and plus 14, they thought that the answer was 6 marbles (a quantity), instead of plus six (a relation). So at first they represented relational statements as statements about quantities, apparently because they did not know how to represent relations. However, after having learned how to represent relational statements, they continued to have difficulties in thinking only relationally, and unwittingly converted the result of operations on relations into statements about quantities. Yet, when asked whether it would always be true that someone who had won 2 marbles in a game would have 2 marbles, the children recognized that this would not necessarily be true. They did understand that relations and quantities are different but they interpreted the result of combining two relations as a quantity.

Unfortunately, Thompson's study does not include quantitative results from which we could estimate the level of difficulty of this type of problem at different age levels but it can be reasonably hypothesized that students at age 13–15 have not yet mastered problems where many relations and quantities must be combined in order to solve the problem.

A brief summary of how students progress in additive reasoning can be gleamed from the literature.

(1) From a very early age, about 5 or 6 years, children can start to use counting to solve additive reasoning problems. They can use the schemas of joining and separating to solve problems that involve combining quantities, separating quantities, or transforming quantities by addition and subtraction.

(2) It takes about two to three years for them to start using these actions in a coordinated fashion, forming a more general part-whole schema, which allows them to solve simple comparison problems.

(3) Combining transformations and relations to solve problems (such as combining two distances to find the distance between two points) continues to be difficult for many students. The CSMS study shows a leveling off of rates of correct responses about age 13; older age groups were not tested in these problems.

(4) The same additive relation can be expressed in different ways, such as "more than" or "less than". When students need to change the relational statement into its converse in order to implement a calculation, they may fail to do so.

(5) Combining positive and negative numbers seems to remain difficult until the age of 14 (no results with 15 year olds were reviewed). The rate of correct responses in some of the problems does not surpass 50%.

Multiplicative Reasoning Problems

Research on multiplicative reasoning problems has produced less agreement in the classification of problem types. The different classifications seem to be based on different criteria rather than on conceptual divergences about the nature of multiplicative problems. We do not attempt to reconcile these differences here but refer to them in footnotes as we describe the development of multiplicative reasoning. We will adopt here Vergnaud's terminology and refer to others as required.

Vergnaud (1983) distinguished between three types of multiplicative reasoning problems:

(1) isomorphism of measures problems, which involve two measures connected by a fixed ratio (Brown, 1981, refers to these problems as ratio or rate);

(2) multiple proportions, in which more than two measures are proportionally related to each other;

(3) product of measures, in which two measures give rise to a third one, the product of the two (Brown, 1981, refers to these as Cartesian problems).[1]

Isomorphism of measures problems include the simple problems described earlier on, which young children can solve by setting items in correspondence. These are the most commonly used type of proportions problems in school; they involve a fixed ratio between two measures. Common examples of such problems are number of people for whom a recipe is prepared and amount of ingredients; number of muffins one makes and amount of flower; quantity purchased and price paid. The level of difficulty of these problems is influenced by the availability of materials that can be used to represent the correspondences between the measures, the ratio between the measures (2:1 and 3:1 are much easier than other ratios), the presence of the unit value in the problem (3:1, for example, is easier than 3:2), and the values used in the problem (if the unknown is either a multiple or a divisor of the known value in the same measure, it is possible to solve the problem using scalar reasoning or within-quantity calculations, the most commonly used by students). In some countries (e.g. France; see Ricco, 1982; Vergnaud, 1983), students are taught a general algorithm (e.g. finding the unit value; the Rule of Three) that can be used to solve all proportions problems, but students often use other methods when proportions problems are presented amongst other problems with different structures (Hart, 1981; Ricco, 1982; Vergnaud, 1983). These student-designed methods have been identified under different terminologies but are remarkably similar across

1 The term *measure* is used here rather than *quantity* because some quantities may be measured differently and problems about these quantities would thus end up in different categories. For example, if the area of a parallelogram is measured with square units, the calculation of its area will be an example of isomorphism of measures problems: number of units in a row times number of rows. If the area is measured using linear units, the calculation is a product of measures, as a square unit such a $1cm^2$ will be the product of the two linear units, 1cm x 1cm.

countries. They involve parallel transformations within each measure (e.g. dividing each measure by two), a move which preserves the ratio between the measures, and a progressive approximation to the answer, without losing sight of the correspondences between measures. Hart's (1981) well known example of the onion soup recipe for 8 people, which has to be converted into a recipe for 6 people, illustrates this approach to solution well. Students tend to calculate what ingredients would be required for 4 people (i.e. half of the ingredients for 8 people), then what would be required for 2 people, and then add the ingredients for 4 with those for 2 people – thus finding the solution for 6 people.

Systematic comparisons using carefully matched values across problems (see, for example, Nunes, Schliemann, & Carraher, 1993) show that the students approach proportions problems more often by thinking of the relations within the same measure than of the relations between the two different measures in the problem. For example, in the same onion soup recipe, the ratio pints of water per person was 1:4. Students could calculate this ratio from the recipe for 8 people and find how much water for 6 people, but this solution was not reported by Hart.[2]

In summary, in the assessment of younger children's competence in isomorphism of measures problems one can vary the level of problem difficulty by varying the materials available for representing the problem; in the assessment of older children, one can vary the level of problem difficulty by using numbers that make either within-quantity or between-quantities calculation easier.

Multiple proportions problems involve a situation in which more than two measures have a fixed ratio. Vergnaud proposed as an example problem the question of finding the amount of milk produced in a farm, which is related to the number of cows in the farm and the number of days. Multiple proportion problems are more difficult than simple isomorphism of measures problems, as they involve handling more information and carrying out more calculations. It is, however, not clear whether they pose new conceptual challenges for students.

2 Nesher (1988) and Schwartz (1988) suggest that dividing one quantity by another, the move required to calculate the ratio of water to people, changes the referent of the number: instead of thinking of 2 pints of water, one must think of 1 pint per 4 people. They attribute to this transformation of the referent the higher level of difficulty of some problems. This leads them to classify multiplication problems using a different schema, which is not discussed here.

Proportional reasoning is one of the most crucial areas of mathematics education, as it is the basis of understanding in several further domains of mathematics. It is applied in other school subjects, first of all, in science. Several everyday life contexts require handling of proportions as well. Therefore, a number of large-scale assessment projects explored its development (Kishta, 1979; Schröder at al., 2000; Misailidou, & Williams, 2003; Boyera, Levinea, & Huttenlochera, 2008; Jitendra at al., 2009). In a cross-sectional assessment where its development was examined in relation to analogical and inductive reasoning, it was found that at grade five only 7% and at grade seven 20% of students was able to solve a simple proportional task (Csapó, 1997).

Product of measures problems involve more a composition of two measures to form a third measure: for example, the number of T-shirts and number of skirts a girl has can be composed to give the number of different outfits that she can wear; the number of different colored cloths and the number of emblems determines the number of different flags that you can produce; the different types of bread and the number of different fillings in a delicatessen can be combined to form different types of sandwiches. Thus product of measures problems involve a qualitative multiplication – i.e. the combination of different qualities in a multiple classification – as well as a quantitative multiplication. Product of measures problems are significantly more difficult than isomorphism of measures problems (Brown, 1981; Vergnaud, 1983). They are a significant part of multiplicative reasoning and thus should be evaluated in the assessment of students' multiplicative reasoning.

The development of students' understanding of product of measures problems is not an all-or-nothing matter. These problems can be simplified in presentation, by using suggestions of how the combinations work in a step-wise presentation: with one skirt and 3 blouses, how many different outfits can you make; if you buy a new skirt, how many new combinations can you make? When product of measures problems are presented in a step-wise manner, the rate of correct responses increases significantly (Nunes & Bryant, 1996). Figure 1.3 shows an example of a product of measures problem in which the first two combinations are presented; students might easily find the remaining combinations, and count the total possible number. Students aged 11–12 years in the U.K. showed an average of 81% correct responses to this problem; this was significantly better than the rate of 51% correct observed in problems in which only one item was used to suggest how the combinations might work.

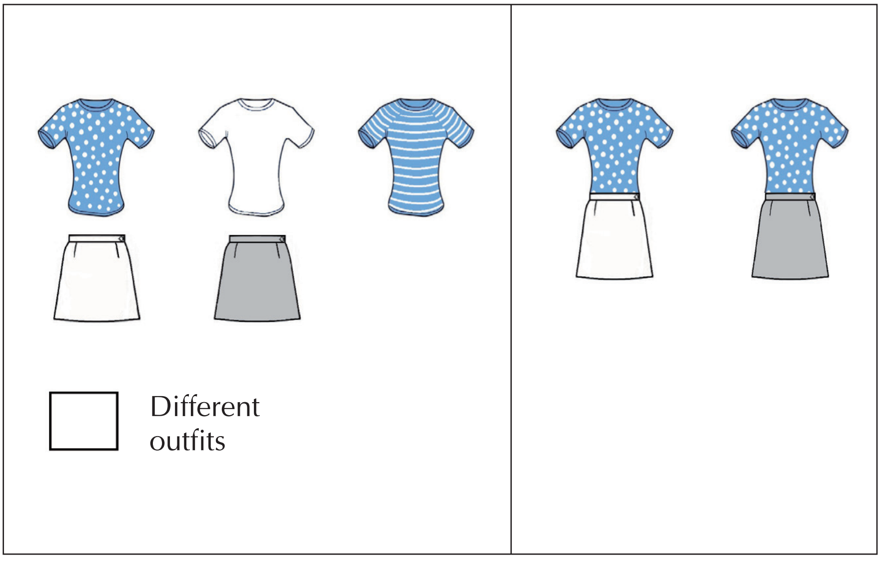


Figure 1.3 An example of product of measures problem in which the first combinations are presented visually

However, students may not necessarily be able to formulate a general rule for finding out the number of possible combinations after this step-wise introduction. The step from thinking that for each new skirt, *x* new outfits to the general rule that, therefore, the number of outfits is the number of skirts times number of outfits, demands considerable effort.

In conclusion, the development of multiplicative reasoning involves two types of correspondences: those exemplified in isomorphism of measures problems, which are quantitative, and those exemplified in product of measures problems, in which an initial, qualitative step based on a multiple classification schema is required. Young children can succeed in isomorphism of measures problems if they can use manipulatives to represent the measures; they solve the problems by counting (i.e. they do not calculate a multiplication) but their reasoning is clearly multiplicative. Product of measures problems are more difficult.

As children progressively master the relational calculations in multiplicative reasoning, they solve a greater variety of problems. However, one challenge remains, even at the end of primary school. Students seem to have greater difficulty in thinking about the between-quantities, functional relation (measured by the rate between variables) than about the within-quantities, scalar relation. Always solving problems using scalar solutions means that students focus only on half of the relationships that are significant in the situation. Teaching that helps students focus on functional relations also helps students make conscious choices of models for problem solving. Students are known to over-use as well as under-use proportional reasoning (e.g. De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock, Verschaffel, & Janssens, 1998; Dooren, Bock, Hessels, Janssens, & Verschaffel, 2004) and also over-use linear relations when asked to represent graphically the relationship between two variables. It is possible that if students became more aware of the functional relations, i.e. the relations between variables, they would be less prone to such errors.

Advancing Cognitive Development Through Mathematics Education

In the preceding sections, we discussed how reasoning and knowledge of numerical systems are inter-related and support each other in mathematics education. Reasoning about quantities is always necessary for understanding how numerical representations work. In this section, the focus is on how good mathematics education can promote a better understanding of relations between quantities and a greater ability to use numbers and other mathematical tools to solve problems.

Research on Advancing Cognitive Development in Mathematics Education

The examples of forms of reasoning about quantities described in the earlier sections are not innate: they develop over time, and this development can be promoted in the context of mathematics education. The influences between mathematical reasoning and learning mathematics in the classroom are reciprocal, in so far as promoting one leads to improvement in the other.

Research by different teams of investigators (Nunes et al., 2007; Shayer & Adhami, 2007) has shown that improving students' thinking about mathematics in the classroom has a beneficial effect on their later mathematics learning. We present briefly here some results from the project by Shayer and Adhami, which included a large number of classrooms and of pupils and extensive professional development for the teachers. Shayer and Adhami's (2007) study included approximately 700 students and their teachers, approximately half of whom were in the control and the other half in experimental classes. The researchers designed a program known as CAME (cognitive acceleration through mathematics education) to be used by grade 5 and 6 teachers and their children (9 to 11 years old), which emphasized reasoning about numerical problems. The teachers participated in two full-day professional development workshops, in which they discussed and re-designed the tasks for their own use. The pupils were assessed in a Piagetian task of spatial reasoning before and after their participation in the program. For the control group, mathematics teaching went on in the business-as-usual format during this period.

In the Piagetian Spatial Relations test (NFER, 1979), children are asked to draw objects in situations chosen to test their notions of horizontality, verticality and perspective. For example, they are asked to draw the water level in jam-jars half-full of water, presented at the various orientations: upright, tilted at 30 degrees off vertical, and on its side. They are also asked to produce a drawing of what they would see if they were standing in the middle of a road consisting of an avenue of trees; the drawing should cover the near distance as well as afar. Assessment of the children's responses consists of seeing how many aspects of the situation, how many relations, they can consider in their drawings. The tasks allow for a classification of the productions as Piaget's early concrete operations (level 2A), if the drawings represent only one relation, or as mature concrete (2B), if they handle two relations.

The tasks included in the program considered many of the issues raised here: for example, with respect to rational numbers, the students were asked to compare the amount of chocolate that recipients in two groups would have; in one group, one chocolate was shared between 3 children and in the other two chocolates were shared between 6 children. The equivalence of fractions could be discussed in this context, which helped the students understand the equivalence in quantities in spite of the use of different fractions to represent the quantities.

Shayer and Adhami (2007) observed a significant difference between the students in the control and the experimental classes, with the experimental classes out-performing the control classes in the Piagetian task as well as in the standardized mathematics assessments designed by the government, and thus completely independent of the researchers.

In summary, mathematical tasks that are well chosen in terms of the demands they place on students' reasoning, and are presented to students in ways that allow them to discuss the mathematical relations as well as the connections between quantities and symbols, contribute to mathematical learning and cognitive development.

Numeric Skills, Additive and Multiplicative Reasoning

The previous sections in this chapter identified different reasoning skills to be developed, as playing a central role in early mathematics education and determining later achievements. This section summarizes the different skills related to this area and outlines their development.

Whole Numbers

In pre-school, children should have the opportunity to learn about the relations between quantities and numbers. The indicators presented here are not exhaustive, but all children must be able to understand that:

(1) if a quantity increases or decreases, the number that represents it changes
(2) if two quantities are equivalent, they are represented by the same number;
(3) if the same number of items is added to and taken away from a set, the number in the set doesn't change;
(4) any number can be composed by the sum of two other numbers;
(5) when counting tokens with different values (money, for example), some tokens count as more than one because their value has to be taken into account.

Children who understand these principles make more progress in learning mathematics throughout the first two years of school than those who do not.

Rational Numbers

Fractions are symbols that represent quantities resulting from division, not from counting. They represent the relation between the terms in a division. Children can start to explore these insights in kindergarten and in the first years in primary school by thinking about division situations. They should be able to understand that:

(1) if two dividends are the same and two divisors are the same, the quotient is the same (e.g. if there are two groups of children with the same number of children sharing the same number of sweets (or sharing cakes of the same size), the children in one group will receive as much as the children in the other group;

(2) if the dividend increases, the shares increase;

(3) if the divisor increases, the shares decrease;

Further insights into the nature of division and fractions can be achieved from about age 8 or 9:

(4) it is possible to share the same dividend in different ways and still have equivalent amounts; the way in which the shares are cut does not matter if the dividend is the same and the divisor is the same;

(5) if the dividends and the divisors are different, the relation between them may still be the same (e.g. 1 chocolate shared by 2 children and 2 chocolates shared by 4 children result in equivalent shares);

(6) these ideas about quantities should be coordinated with the writing of fraction symbols.

These insights about rational numbers enable students to use rational number to represent quantities and can be used to help them learn how to operate with numbers. However, in order to solve problems, students need to learn in primary school to reason about two types of relations between quantities, which lead to mathematizing situations differently: part-whole, which define additive reasoning, and correspondence relations, which define multiplicative reasoning.

Additive Reasoning

The development of additive reasoning involves a growing ability to distinguish quantities from relations and to combine positive and negative relations even without knowledge of quantities. Although there is no single investigation that covers the development of additive relational reasoning, a summary of how students progress in additive reasoning can be abstracted from the literature.

Level 1 Students can solve problems about quantities when these increase or decrease by counting, adding and subtracting. They do not succeed in comparison problems.

Level 2 Students succeed with comparison problems and also in using the converse operation to solve problems. The same additive relation can be expressed in different ways, such as "more than" or "less than". When students need to change the relational statement into its converse in order to implement a calculation, they may fail to do so. At level 2, they are able to convert one relation into its converse in order to solve problems.

Level 3 Students become able to compare positive and negative numbers and to combine two relations to solve problems, but they often do so by hypothesizing a quantity as the starting point for solution. Combining more than two positive and negative relations in the absence of information about quantities remains difficult until the age of 14 (no results with 15 year olds were reviewed). The rate of correct responses in some of the problems does not surpass 50%.

Level 4 Perfect performance in combining additive relations and distinguishing these from multiplicative relations.

Multiplicative Reasoning

Multiplicative reasoning starts with young children's ability to place quantities in one-to-many correspondences to solve diverse problems, including those in which two variables are connected proportionally and sharing situations. It involves the understanding of the notion of proportionality, which includes situations in which there is a fixed ratio between two variables in isomorphism of measures problems, and understanding the multiplicative relation between two measures, which can be combined to form a third one, in product of measures problems.

Level 1 Students can solve simple problems when two measures are explicitly described as being in correspondence and they can use materials to set the variables in correspondence. However, in more complex situations, in which they need to think of this correspondence themselves, they realize that there is a relation between the two variables, so that a change in one variable results in a change in the other one, but may not be able to think of how to systematically establish correspondences between the variables.

Level 2 Students at this level recognize that the two values of the two variables vary together and in the same direction and there is a definite rule behind co-varying. In simple cases and familiar contexts, they recognize the quantitative nature of the relationship but are unable to generalize a rule.

Level 3 At this level students recognize the linear nature of the relationship, and they are able to deal with proportionality in familiar contexts.

Level 4 At this level, students are able to deal with the linear relationship of the two variables in any content and context. They are also able to distinguish linear from non-linear relationships, although they may need to make step by step comparisons when asked to think about novel problems.

The hierarchy outlined here corresponds to the natural order of cognitive development. If teaching always focuses on the level next to the already reached level – individually in cases of every student –, then they possess the mental tools needed for comprehending it. In this way teaching may have optimal developing effect.

Further Areas for Advancing Mathematical Thinking

Beyond the areas of mathematical reasoning discussed in the previous parts of this chapter, there are several further ones to be developed in the early mathematics education. We review some of them in this section, but we do not deal with them in detail. Although the areas of mathematical reasoning are related to each other, the areas of reasoning reviewed in this section are not directly related to numerical reasoning or they are generalized beyond the issues of numbers. Furthermore, fostering their development may also be possible by exercises embedded in other school subjects; therefore, the advancement of reasoning abilities reviewed here may not be narrowed to mathematics education. For example, text comprehension assumes understanding and interpreting operations of propositional logic. Processing complex scientific texts, especially comprehending sophisticated definitions requires handling logical operations. Learning science activates a number of cognitive skills which are developed in mathematics. In this way science education enriches the experiential basis of mathematical reasoning in several aspects, such as seriations, classifications, relations, functions, combinatorial operations, probability and statistics.

Most reasoning abilities listed here were extensively studied by Piaget and his followers. According to their findings, the development of these schemes begins early, well before schooling starts. In the first six years of schooling, in which we are interested, their development is mostly in pre-operational and concrete operational phase, and the formal level can be reached only in the later school years. Therefore, the main task of early mathematics education is to provide students with a stimulating environment to gain experiences for inventing similarities and rules to create their own operational schemes. These systematic experiences should be followed by mastering the mathematical formalisms later. Science education, especially hands-on-science, may contribute to the development by enriching the experiential bases in the early phase, and later at the higher level of abstraction by the application of mathematical tools.

Logical operations and the operations with sets are isomorphic from the mathematical point of view, but the corresponding thinking skills are rooted in different psychological developmental processes. However, their similarities may be utilized in mathematics education. The development of logical operations was examined in detail by Piaget and his co-workers in their classical experiments (Inhelder & Piaget, 1958). Later research has indicated that not only the structure of logical operations determines how people judge the truth of propositions connected by logical operations but the familiarity of context and the actual content of propositions as well (see Wason, 1968, and further research on the Wason task). However, the aim of mathematical education is to help students to comprehend propositions and interpret their meaning determined by the structure of the operations, therefore the conclusions of Piaget's research remain relevant for mathematics education. Furthermore, Piaget's notion that development takes place through several phases and takes time should also be taken into account. As for the operations with sets, for which several tools are available for manipulation, may serve as founding experiences for logical operations. The schemes of concrete, manipulative operations carried out by objects may be interiorized and promote the development of operations of propositional logic. On the other hand, developing propositional logic is a broader educational task, in which pre-primary education should play an essential role, as well as several further school subjects. In the later phases teaching of other school subjects may contribute to fostering the development by analyzing the structure of logical operations and by highlighting the relationships between structure and meaning. There are several broadly used instruments for assessing the development of logical operation (see e.g. Vidákovich, 1998).

Relations appear in several areas of mathematics education. Reasoning with some relations has been discussed in the previous sections, and several further operations involving relations were examined by Piaget, too (Piaget & Inhelder, 1958). Among others these are seriations and class inclusions. The construction of series plays a role in the development of proportional reasoning discussed earlier and may contribute to several broader reasoning abilities, such as analogical and inductive reasoning (see Csapó, 1997, 2003). Recognizing rules in series and correspondences in classifications develop skills of rule induction and contribute to the concept of mathematical function. As the mathematical conception of function is a result of multiple abstractions, a solid experiential base in essential for further learning. Relations may be represented and visualized in several ways. Understanding the correspondence between different representations and carrying out transformations between representations may foster analogical reasoning. Developing the skills related to multiple representations is also a task of mathematics education.

From a mathematical point of view, combinatorics, probability and statistics are closely related, but the corresponding psychological developmental processes originate from different points. Spontaneous stimuli coming from an average environment cannot connect these different ideas; only systematic mathematics teaching may lead to establishing connections among them at a more mature level.

Combinatorial problems may be classified into two main groups. In enumeration tasks students are expected to create all possible constructs out of given elements, permitted by conditions or situations. Some problems of this type may be solved by combinatorial reasoning. In the other group are the computation problems, when the number of possible constructs should be calculated, which, in general case, can be solved only after systematic mathematics education. We have already discussed some aspects of combinatorial reasoning concerning the multiplicative reasoning. Combinatorial structures play a central role in Piaget's theory of development of propositional logic, and he also examined the development of some combinatorial operation (Piaget, és Inhelder, 1975). Several further research projects explored the structure and development of combinatorial thinking and the possibilities of fostering it both in mathematics and in other school subjects. (Kishta, 1979; Csapó, 1988, 2001, 2003; Schröder, Bödeker, Edelstein, & Teo, 2000; Nagy, 2004). An analysis identified 37 combinatorial structures, according to the number of variables, the number of values of variables, and the number of constructs to be created that may be handled. On the basis of these operational structures, test tasks can be devised. The empirical investigation based on these tasks revealed that some children were able to solve every task by age 14, but most of them were able to deal only with the most basic operations (Csapó, 1988). The charge of early mathematical education is the stimulation of the development of combinatorial reasoning by well structured exercises, while enumeration tasks may be embedded in other school subjects as well, which can also foster combinatorial reasoning (Csapó, 1992). Nevertheless, preparing the formalization of reasoning processes and teaching computational problems can be done only in mathematics.

The development of the idea of chance and probability begins early (Piaget & Inhelder, 1975), but without systematic stimulation most children reach only a basic level. Understanding nondeterministic relationships and correlations is especially difficult (Kuhn, Phelps, & Walters, 1985; Bán, 1998; Schröder, Bödeker, Edelstein, & Teo, 2000). Development of probabilistic reasoning may be promoted in early mathematics by illustrating chance, while other school subjects (e.g. biology) may present probabilistic phenomena to enrich the experiential basis of development. Later, systematic exercises may prepare the introduction of formal interpretation of probability as ratio of the number of occurrences of different events.

A further area, spatial reasoning is rooted in other developmental processes, different from that of numeric reasoning, and is related to measures and numbers in later developmental phases in the framework of systematic geometry education. Piaget explored the development of spatial reasoning mostly trough the representation of space in children's drawings. According to his results, early development may be characterized by a topologic view, when first the connecting points of lines are correct on drawings, but shapes are distorted. The shapes drawn by children get further differentiated during the second stage (age 4–7). In the third stage children draw shapes and forms which are correct in Euclidean terms (Piaget & Inhelder, 1956). Spatial reasoning may be fostered in the early mathematics education by systematic exercises of studying two and three dimensional forms. Then, students may be encouraged to infer that shapes have properties, and similarities and differences between shapes may be characterized by these properties. Later, properties may be precisely defined in the framework of geometry teaching.

Spatial reasoning may be fostered in the framework of teaching drawing and art education as well. A number of different instruments have been devised for the assessment of representation of space in the framework of art education (see e.g. Kárpáti & Gaul, 2011; Kárpáti & Pethő, 2011).

Assessing Cognitive Development in Mathematics

One of the major points in the preceding discussion was the significance of reasoning for understanding number system and for understanding how to use mathematics to model the world. A second thread in the discussion is that most insights that students have to develop in mathematics do not develop in a single step. For this reason, assessments of cognitive development in the context of mathematics should be designed in ways that place little demands on computation in comparison to relational calculations. Computation can, and should, be assessed on its own merits.

Content of Assessment

Reasoning skills that are predictive of mathematics achievement must be at the core of assessments of cognitive development in mathematics. Different predictive studies have shown that, in the early years, children's performance in tasks that assess their knowledge of correspondences, seriation, additive composition, and the inverse relation between addition and subtraction predict their performance later, in standardized tests of mathematics achievement (Nunes et al., 2007; van de Rijt, van Luit, & Pennings, 1999). Early number skills, sometimes referred to as number sense, is also a predictor of mathematics achievement (Fuchs, Compton, Fuchs, Paulsen, Bryant, & Hamlet, 2005).

Measures of early number knowledge include knowing how to write and read numbers, how to compare the magnitude of written and oral numbers, and some computation bonds. As far as we know, only one study (Nunes et al., 2011) has compared the relative importance of number skills and mathematical reasoning in the prediction of mathematical achievement in a large scale longitudinal analysis. This comparison also included the cognitive functions of attention and memory in the analysis, and a measure of IQ. All predictors were assessed when the students were in the 8–9 year age range; the measures of mathematical achievement were independent standardized measures obtained by the school when they were 11–12 and 13–14 years. All the predictors made significant and independent contributions to variation in mathematics achievement at ages 11–12 and 13–14. For both time points, mathematical reasoning made a stronger contribution than attention and memory and also than numerical skills. This analysis of the relative importance of mathematical reasoning and number skills suggests that, if time limits are significant, it is more important to assess mathematical reasoning than numerical skills.

Forms of Assessing Thinking Abilities in Mathematics

The design of mathematical assessments is inevitably related to students' general ability to understand instructions and other verbal skills. However, it is possible to minimize the influence of reading skills by designing assessments that use drawings to help students imagine the problem situations. Drawings also allow students to use different approaches in establishing the numerical value of their answer: they can often analyse the drawings (e.g. divide something in two to help them imagine the value of half the quantity) and even count in order to determine the answer. As long as their analysis of the situation is correct, they have a better chance of quantifying the answer correctly than if the problems were presented only in writing. Researchers in the Freudenthal Institute pioneered this approach to assessment (see, for example, van den Heuvel-Panhuizen, 1990), which gives valuable information about students' relational calculation skills.

As discussed earlier on, mathematical insights develop over time. Ideally, one should include in assessments different levels of demands on the reasoning skills. These can be varied by using different forms of representation, different situations, as well as different values. The preceding review suggests how these variations can be attained within the assessment of the same type of concept.

Summary

In this chapter we have reviewed some major areas of the development of mathematical reasoning. We have focused our attention on those psychological questions which are the most crucial ones from the point of view of early cognitive development. We highlighted those thinking abilities that may be developed almost exclusively in the framework of mathematics education. Among these areas are the reasoning about measures and numbers and the development of the relations among concepts and skills.

We have emphasized that developing mathematical thinking differs from mastering mathematical knowledge. The beginning phase of schooling should focus on fostering the development of mathematical thinking, as without proper reasoning skills mathematics cannot be comprehended.

We have discussed four areas in more detail: reasoning about whole numbers, rational numbers, additive and multiplicative reasoning. These are especially important as they form the foundations for later mathematics learning. Results of several research projects indicated the predictive power of these thinking abilities; the early levels assessed in these areas predict the achievements measured later.

We have also indicated that there are several further important components of mathematical reasoning. They can be developed and assessed in similar ways to those that were described in more detail.

We have emphasized at several points that the development of mathematical thinking is a slow and long process taking several years. However, several research and development projects have shown that systematic stimulating exercises can accelerate development. These exercises can result in improvement of thinking only if they are carefully matched to the actual developmental level of students. Therefore, in mathematics education, the sequence of developmental stimuli is especially important. A complex thinking process can develop successfully only if the preceding phase has already been completed and the component skills developed.

Consequently, for the development of mathematical thinking, it is inevitable that teachers know well the actual developmental level of their students. This allows them to adjust teaching individually to the need of every student. In order to meet this need, the instruments of diagnostics assessment should cover the developmental process of mathematical thinking.

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2

Mathematical Literacy and the Application of Mathematical Knowledge

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One of the most important sources of objectives for learning mathematics can be summarized as the needs coming from the society in general and from other disciplines, especially sciences. Therefore, mathematics as a discipline and as a school subject may shape students' minds in a way that they develop a disposition to use their mathematical knowledge in several different contexts including other school subjects and everyday out-of-school problem contexts.

The idea of describing how the mathematical knowledge achieved in schools can be applied in various contexts and problem spaces is at least of the same age as the emerging mathematical ideas. Therefore the general theoretical fundamentals of the application phenomena will be first shortly presented in this chapter. In the last centuries, in most European school systems mathematics as a school subject earned the position of having a central role in curricula. Since the Ratio Studiorum, when Christopher Clavius exerted his influence on making mathematics a standard part of the Jesuit core curriculum (see Smolarski, 2002), till today's core curricula in Europe, there is a continuous search for better ways in teaching and learning mathematics. The second part of this chapter will focus on some assessment considerations about the applications of mathematics.

In the third part of this chapter, the characteristics and role of classroom mathematics tasks will be analyzed with a special emphasis on word problems. It is the classroom practice and culture that shape students' beliefs about and approaches of different types of word problems. Finally, we aim to provide a categorization of mathematical word problems in view of developing a diagnostic evaluation system of mathematical literacy.

Theoretical Considerations

In the history of mathematics and mathematics teaching there were continuous attempts and efforts made in order to bring evidence about the importance of mathematics in everyday life and in other sciences. These efforts have often been hindered by the dual nature of mathematics, i.e., the way mathematical results were published and communicated, and the way mathematical thinking and explorations have been actually performed.

The Nature of Mathematical Thinking

Mathematics is often associated with creating theorems, proofs and definitions. From ancient times, mathematical publications followed strict rules in presenting mathematical results. These rules are essentially the rules of deductive implications. The structure of many mathematical publications even today follows the sequence of definition – theorem – proof. However, as early as in the seventeenth century Descartes claimed that the ancient Greeks in fact yielded their theorems in an inductive way while they published their results according to strict deductive rules. The duality of how theorems are presented and how they have been achieved can even confuse laymen who often consider mathematician as people who create theorem and prove them. Nevertheless, Rickart (1996) emphasizes - following in Poincaré's and Hadamard's footsteps – that creativity plays an essential role in mathematical discovery. Conscious hard work and creative experiences go in tandem when doing mathematics. Although different facets of mathematical thinking go in tandem in doing mathematics, one or the other may noticeably appear, depending on the task to be solved. "Even inside the profession we classify ourselves as either theorists or problem solvers." (Guy, 1981, p. vii.) Ernest (1999) suggests keeping a balance between explicit propositional and tacit mathematical knowledge in educational contexts.

The key for understanding how school mathematics reflects different philosophical approaches can be found in Freudenthal's oeuvre. What students should learn in schools is to do mathematics and not primarily to accept the products of (mathematicians') mathematical activity. Doing mathematics requires students to gather experiences, form hypotheses, and above all, to learn to think mathematically. "The learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulae; algorithmising rather than algorithms; verbalizing rather than language… (Freudenthal, 1991, p. 49). Contrary to the historically developed DTP order (definition – theorem – proof), for mathematics lessons a reversed order should be applied: exploration, explanation, formalization (Hodgson, & Morandi, 1996).

A Mathematical Modeling Perspective

"The emergence of the discipline Mathematics Education in the beginning of the 20th century had a clear political motivation" (Sriraman & Törner, 2008, p. 668.) The main supporters of different movements were of economic nature. There are two mathematics education movements in the twentieth century that have strong influence on the principles and practices of even today's mathematics education. The New Math movement aimed at emphasizing mathematical structure through abstract concepts. Following the works of the Bourbaki group, the New Math movement has resulted in highly formalized textbooks, and initiated school curriculum and teacher education reforms. The New Math movement emphasized the *why*s and the deeper structure of mathematics, instead of mindless rigidity of traditional mathematics (Sriraman & Törner, 2008). That is why it is worth evaluating that movement in a more positive way instead of merely criticizing it from a postmodern math education perspective. This movement initiated studying the similarities between mathematical and psychological (hypothetico-deductive) structures as well.

The Realistic Mathematics Education (RME) movement is "a reaction to both the American New Math movement … and the then prevailing Dutch

… 'mechanistic mathematics education." (van den Heuvel-Panhuizen, 2001, p. 1). The RME grew out of Hans Freudenthal's initiations: founding the Wiskobas project (in Dutch: 'mathematics in primary school') and later the Freudenthal Institute, and at the same time fertilizing mathematics education with ideas such as that student should develop and apply concepts and tools for daily life problem situations that are meaningful for them (van den Heuvel-Panhuzien, 2003). As already indicated in the above-mentioned quotation from Freudenthal, realistic mathematics educations aims at the construction by children of their own mathematical knowledge, emphasizing human activity as mathematizing both within the mathematical structure and between learned knowledge and context situations (see Treffers, 1993; Wubbels, Korthagen & Broekman, 1997). Since in English and in other languages the translation of the term 'realistic' will be associated with 'reality' there were attempts to clarify how reality and realistic should be defined in mathematics education settings (Greer, 1997; Säljö, 1991a, 1991b). As van den Heuvel-Panhuizen (2001a) emphasizes the original Dutch term 'zich realizeren' means 'to imagine', therefore realistic mathematics does not always has the real world as context for tasks; objects of the fantasy world (which can be imagined, represented, and therefore modeled) can form an equally appropriate context for mathematization. The current interpretation of the term 'realistic' is a reference to what is *experientially* real (Gravemeijer & Terwel, 2000; Linchevski & Williams, 1999), declaring that not every everyday-life problem will be necessarily experientially real for the students.

Even though there are signs that there was greater emphasis on links to reality fifteen years ago than there is now in the research and development work of RME (see van den Heuvel-Panhuizen, 2000), the strong and relevant connections between real-life contexts and students' mathematical learning is still a major characteristic of RME. Treffers (1993) developed the concepts of horizontal and vertical mathematization. The term mathematization was developed by Freudenthal (see van den Heuvel-Panhuizen, 1996, 2000, 2001a, 2001b, 2003). Mathematization refers to the processes of mathematical activity; since it is not mathematics as a closed system that should be taught in school, but rather the activity of organizing matter from reality. Treffers' horizontal mathematization concept refers to the process of bringing mathematical tools forward in order to organize and solve daily life problems. Vertical mathematization refers to inner mental reorganization of concepts and operations within the mathematical system. Horizontal and vertical mathematization processes are intertwined in students' mathematical activities, and mathematization "contains, in fact, all of the important aspects of the RME educational theory" (van den Heuvel-Panhuizen, 1996, p. 11.).

One crucial point in RME is introducing mathematical models (in a very broad sense of this word). Creating and developing models *for* problem situations is very different from searching for models *of* problem situations (see van den Heuvel-Panhuizen, 2001a). Effective use of several models in different age-groups and in different content areas has been evidenced. Gravemeijer (1994) investigated the empty number line as a powerful mathematical model for several reasons. By means of visualization it enables for using and explaining various strategies, e.g., subtracting 49 can be substituted by subtracting 50 and adding one, or in case of subtracting a relatively large number (e.g., 51 – 49) it may be easier to step forward from the smaller quantity to the larger quantity.

Klein, Beishuizen and Treffers (1998) added that it is not the empty number line alone that contributes to the success of their development program, but the way it was used, i.e. stimulating and discussing different solution patterns in a positive classroom climate. Keijzer and Terwel (2003) studied the understanding of fractions, and also successfully used the number line model (also by means of computer games) to develop understanding. Doorman and Gravemeijer (2009) conducted an experiment among 10th grade students in the field of velocity problems, using discrete graphs as models for reasoning about the relation between displacement in time intervals and total distance traveled. An extension of the RME principles to higher school grades had been previously demonstrated by Gravemeijer and Doorman, (1999) in the field of calculus. In that case determining velocity from time/interval graphs became a model for reasoning about integrating and differentiating arbitrary functions. Van Garderen (2007) argues that diagrams as mathematical models provide the flexibility for children with learning disabilities to generalize what they have learnt in a given situation to another situation.

The realistic mathematics approach proved to be useful also for low attaining students. The principles and suggestions concerning RME for low attaining students have been reviewed by Barnes (2005). Low attaining students and even special education need students profited more from so-called guided instruction, i.e., when much more space is provided for individual contributions, than from a so-called structured or direct instructional approach (Kroesbergen & van Luit, 2002). However, in general, the relationship between mathematical instructional approaches (namely, traditional and realistic approaches) and mathematical proficiency has not been unequivocally evidenced. In general, there are larger differences in pupil performance within a particular mathematics instructional approach than between two different approaches (Koninklijke Nederlandse Akademie van Wetenschappen, 2009).

The Curricular Shaping of Mathematical Literacy

Scientific discourse on the role and importance of curricular aims and objectives has recently been permeated by a range of different curricula as defined according to different levels or phases of the teaching-learning process. When analyzing research-based curriculum development, Clements (2008) narrows the term to available curriculum, i.e. curriculum for which teaching materials exist. There is a usual trinity of curriculum terms used in the (mathematics) education literature: declared, implemented and achieved curriculum. The declared curriculum refers to educational documents set out in different levels of the educational system: national core curriculum, local curricula etc. The implemented curriculum refers to the processes actually carried out in schools, and achieved curriculum refers to students' performance on tests measuring curricular objectives.

In Stein, Remillard and Smith (2007), a diagram shows the relationships between curriculum-related variables including student learning. Although the sequence of the above-mentioned three curricular concepts is straightforward, how these concepts can transform into each other can be explained by several factors. Figure 2.1 also points to the complexity of factors explaining the transition between curricular concepts, listing mutually and necessarily intertwined phenomena as teachers' beliefs, teachers' professional identity, and higher system-level variables as organizational and policy aspects.

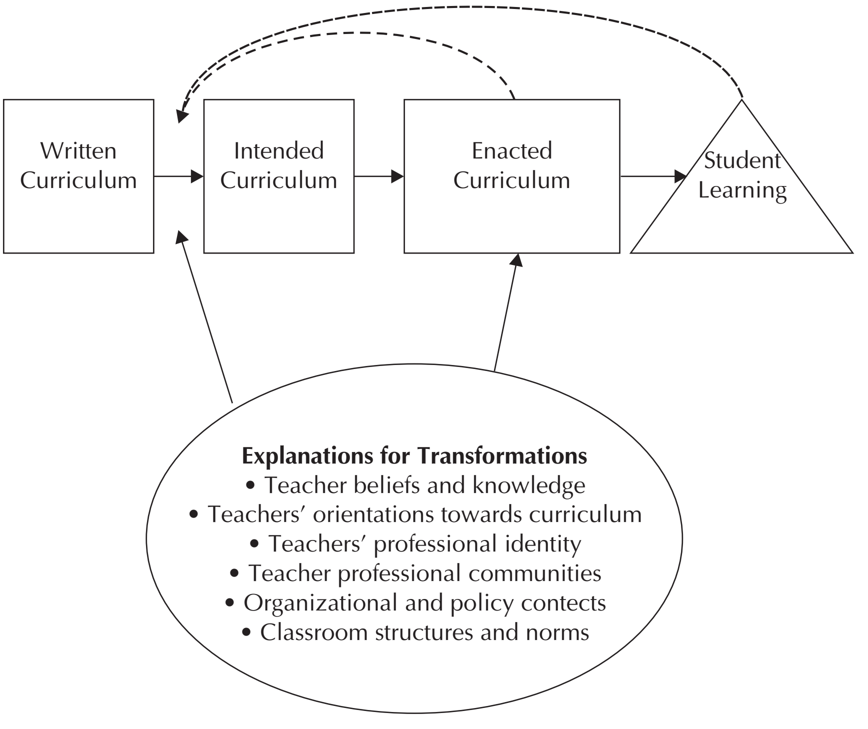


Figure 2.1 Relationships between written, intended and enacted curricula, and student learning (Source: Stein, Remillard & Stein, 2007, p. 322)

Some mathematics task-related factors concerning curricular shaping of knowledge are discussed in Henningsen and Stein's (1997) study. There are at least two steps in between the tasks formed on the basis of the declared curriculum and students' learning outcomes (i.e., the achieved curriculum). Mathematics tasks are set up by the teachers according to their implemented curriculum, and mathematics tasks are in a further step implemented by the students in the classroom. The transition between teacher and student implementations as mentioned in the previous sentence is influenced by several factors including general classroom norms and content-specific sociomathematical norms (Yackel & Cobb, 1996), and teachers' instructional dispositions. The importance of teachers' beliefs and instructional dispositions will be illustrated in the chapter part entitled "tasks measuring mathematical literacy in the classroom".

In this section we focus on examples from national (declared) curricula, since in some way or another, through several direct and indirect factors, national curricula have their impact on both implemented and achieved curricula. The following examples express how in the last decades our curricula declared and emphasized the importance of approaching classroom-based mathematical knowledge and the mathematical knowledge that is transferable to different types of problems and to other school subjects.

Characteristics of Core Curricula in Mathematics

Before introducing the current National Core Curriculum, the so called "National Curriculum '78" had great impact on the Hungarian school system not only because of its descriptive nature (this national curriculum was compulsory for every schools and there were no local curricula) but to the progressive changes it introduced – among others in the field of mathematics. The mathematics part of the national curriculum followed the structure of other parts of the curriculum, i.e. there were aims, objectives and contents formulated for grades 1–4 and grades 5–8, but C. Neményi, Radnainé and Varga (1981) defined overarching intervals for curricular objectives: the divisions of grades 1–3 and grades 4–5 expressed their beliefs that the necessarily continuous developmental processes in students' mathematical thinking should not be separated into two formally distinct stages at the end of the fourth grade (which is a formal dividing line in Hungarian educational system between lower and upper grades of the primary school).

Among the general objectives of the National Curriculum '78 we found motivation in the sense that students are expected to be interested in, and be fond of mathematics both because of external reasons like utility and applicability and because of internal reasons like harmony, truth and beauty in mathematics. (p. 262). According to Aiken (1970), attitudes towards mathematics in adulthood are determined by childhood experiences, and grades 4 to 6 are of crucial importance in forming attitudes. In Hungary, a nationwide analysis revealed that students' attitudes towards mathematics are of mediocre level (Csapó, 2000).

Other curricular objectives present in the National Curriculum '78 pay special attention to student characteristics of a cognitive nature. As for the application of mathematical knowledge in different context, the following objectives were formulated.

In grade 4 and 5 "judgments about (discussion and defending of) unambiguity of tasks, whether a task contain redundant data, incoherent conditions, and whether a given solution process is suitable." (p. 262.) Among the more concrete objective that are connected to a given grade, in grade 5, we found "ability to determine what data are redundant, and what data should be presented in a word problem", an objective that usually (albeit implicitly) implies horizontal mathematization processes. By the end of grade 3, students are required to "be proficient in gathering and organizing data of a word problem. Students must be able to find an appropriate mathematical model (drawing, displaying, operations, open statements), and to solve a word problem by means of that model or by means of trial and error" (p. 283.) The latter objective more explicitly refers to the need of horizontal mathematization in word problem solving.

The National Core Curriculum (Nemzeti alaptanterv; first version: 1995, latest version: 2007) leaves more space for school autonomy, and formulates nationwide curricular objective more loosely and more generally. It is the local curricula that have to elaborate the general nationwide curricular aims and objectives. In line with current trends in international system-level survey requirements, the definition of 'mathematical competence' contain as important element that "the individual is able to apply basic mathematical principles and processes in acquiring knowledge and in solving problems in daily life, at home and at the workplace." (p. 9.) Most of the age-related objectives in the National Core Curriculum are attached to more than one – two year long each – age intervals.

The structure of the NCC objectives follows the two year long interval scheme, i.e. the first milestone in objectives is the end of the second grade, the second milestone is the end of fourth grade etc. The second aspect of the curricular objectives in NCC is the sub-domains of mathematical literacy. One of the sub-domains is labeled as "Application of knowledge". This sub-domain contains curricular objectives explicitly referring to daily life situations and other school subjects. The objective of applying mathematical knowledge in daily life situations is prescribed from the third age cohort (i.e., from grade 5) to grade 12 throughout all grades. The current evaluation framework may and should address the importance of this objective from as early as the first grade of schooling. The relation between knowledge acquired in the classroom and possible applications in real life situations should be strengthen by means of both instructional and evaluation methods.

As Hiebert et al. (1996, p. 14.) warns, "the tension between acquiring knowledge and applying it is not special to mathematics". "The separation of school learning from 'everyday life' has become a problem receiving significant attention by researchers focusing on the sociocultural nature of cognition" (Säljö, 1991a). However, according to Hiebert et al., an emphasis on the application dimension of knowledge may result in less predictable curricula and teachers may worry about the loss of important information, i.e. not covering some parts of the curriculum because of working with time consuming application tasks. The characteristics and problems of math teacher education cannot systematically be reviewed here, albeit some features are highlighted by Szendrei (2007) who reviewed tendencies and efforts in Hungarian mathematics education and mathematics teachers education research from 1970. One of her most important suggestions is that in math teacher training more time should be dedicated to the didactics of mathematics – currently much stronger emphasis is put on the teaching of mathematics itself.

Applications of and Demand on Mathematical Knowledge in other School Subjects

Historically, mathematics fulfilled a leading role in the development of sciences. As Maddy (2008) expresses, till the seventeenth century, great thinkers of those times could not separate mathematics and science. It was the nineteenth century when mathematicians began to develop concepts that had no direct physical meaning. The historical development of mathematics and sciences still has its effects on school curricula and on classroom practice. Interestingly, the Hungarian National Core Curriculum (Nemzeti Alaptanterv, 2007) does not explicitly mention the terms mathematics or mathematical when detailing the learning objectives of the cultural domain "Man and nature". However, within the cultural domain "Our Earth and environment", there are several points in which the role of mathematical abilities (competencies) in geographical and environmental knowledge acquisition is emphasized. There are three main clusters described in which the importance and role of mathematics can be understood: (1) numerical skills for measurements and data handling, (2) spatial intelligence for spatial orientation and (3) logical reasoning, especially in understanding complex spatial and environmental systems.

In sum, there are unexpectedly few explicit relations between mathematics and science objectives in the Hungarian NCC. Of course, there are connections made by teachers between science topics and mathematical prerequisite knowledge, but Pollak's (1969, p. 401) older critical comment that "the student is typically not given the opportunity to participate in making the abstraction from the physical reality to the mathematical model" still applies to the current classroom practice. Some changes are expected to appear in the near future, in part due to the Rocard-report (High Level Group on Science Education, 2007) on inquiry-based learning and the projects just have started like PRIMAS (Promoting Inquiry in Mathematics and Science Education).

The Definition of Mathematical Literacy in the PISA Studies

The PISA (Programme for International Student Assessment) studies aim at defining and measuring students' knowledge and skills in important areas as mathematical, reading and scientific literacy. It was the PISA 2003 study that focused on mathematical literacy (OECD, 2004). This document emphasizes that the "literacy approach" expresses the intention to define and assess mathematical knowledge and skills not in terms of mastery of the school curriculum, but in terms of readiness for full participation in society.

Based on the more general economic definition of "human capital", the PISA studies define mathematical literacy as follows (OECD, 2003, p. 24):

> "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen."

The components of this definition are further elaborated in the above-mentioned document, e.g., the term "world" refers to natural, social and cultural objects, and it is further clarified by referring to Freudenthal's oeuvre. The system of the PISA mathematical tasks is based on the above definition of mathematical literacy. Students have to solve tasks belonging to different content, process and context dimensions. Consequently, the criterion of "use and engagement with mathematics" points to the need of mastering mathematical knowledge applicable in different content domains, on different competency levels and in different contexts. The term "reflectivity" calls forth building awareness and meta-representations fostering knowledge transfer processes across domains (Adey et al., 2007).

The importance of the PISA studies and the further possibility of using their results in evidence-based policy making has been convincingly evidenced by several secondary analyses (e.g., see Baumert et al., 2009).

Tasks Measuring Mathematical Literacy

In this section we analyze how classroom tasks of mathematical literacy are used and what characteristics they have. From an educational evaluation point of view, tasks of formative evaluation will be discussed, i.e. tasks that are embedded in the teaching-learning process in order to develop students' mathematical understanding. We focus on tasks of mathematical literacy where the definition of mathematical literacy is taken from the PISA studies. With regard to the application-related objectives of mathematical knowledge, the context dimension of PISA can be understood as the application of mathematical knowledge in different situations (OECD, 2006).

The PISA literacy approach (OECD, 1999) requires students "be involved in the full mathematical modeling cycle" (Palm, 2009, p. 3), solving tasks that address even out-of-school settings. Although the PISA mathematical literacy has been worked out for measuring 15 year old students' achievement, as we would like to emphasize, even young children's mathematical literacy can be improved and measured in different contexts, in different fields of application.

Characteristics of Classroom Mathematics Word Problems

In this section we restrict our analysis of mathematical tasks that are relevant from the aspect of application of mathematical knowledge. Since the application of mathematical knowledge usually requires the use of textual elaboration (at least in the phase of posing the problem), word problems will be in the focus of our analysis.

"Word problems can be defined as verbal descriptions of problem situations wherein on or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement." (Verschaffel, Greer & De Corte, 2000, p. ix.)

Historically, word problems fulfilled two interfering roles during the last several centuries. From as early as ancient river valley civilization times, mathematical word problems provided the means for mastering arithmetical skills and at the same time providing tools for solving daily life problems that were of crucial importance in a certain historical context. Work of ancient Egyptian workers or computations necessary to be a successful Venetian merchant required both high-level arithmetical skills and strong connections between problems arisen from daily life and between mathematical prototype examples (see Verschaffel, Greer & De Corte, 2000). This duality of the functions of word problems has lived on till today, and the interference and the state of being intertwined result in questions about the effective use of word problems in classrooms.

The importance of word problems in improving the applications of mathematics has been justified by Pollak (1969, p. 393) in the following way: "How does the student become involved in applications of mathematics? Throughout most of his education, mainly through … 'word' problems".

Types of classroom mathematical word problems may be grouped and analyzed according to textual, semantic and mathematical features they have. Educated people can easily distinguish among different types of word problems. As Säljö (1991b) pointed out, even the twentieth century reader can easily recognize the genre of a mathematical word problem text, and may be capable to handle texts like the following one from 1478:

> *If 17 men build 4 houses in 9 days, how many days will it take 20 men to build 5 houses?*

As long as the solver knows that there exist a direct proportional relation between the number of men at work and the number of houses being built, "our familiarity with this genre leads us to recognize that the extra-linguistic activity that is being referred to – building houses – is, if not accidental, at least not central to the task as an exercise in elementary arithmetic." (Säljö, 1991b) The content of this task can be varied without restraint, and it is not necessary to know any house-building technologies or team working method to solve the task. What is more, it would be disadvantageous to start a deep semantic analysis of the reality of task variables. "The pseudo-real contexts … encourage students to see school mathematics as a strange and mysterious language" (Boaler, 1994, p. 554.). The micro-worlds of word problems (this term is borrowed from Lave, 1992) belong to the same genre of texts, a genre that was caricatured two centuries ago by Flaubert writing his letter about the ill-famed 'How old is the captain?' problem.

Boaler (1994) criticized the so-called pseudo-real type of mathematics word problems from a feminist point of view. Although many tasks are equally strange for both boys and girls, in Boaler's research girls suffered more from pseudo-real context tasks in traditional learning environments than boys. In her own intervention studies, this traditional approach for ignoring the role of content is seriously challenged and uncovered. The main problem concerning the context of school mathematics word problems is suspending reality and ignoring common sense due to entering the genre of word problem texts. According to Boaler (1994), this difficulty can be overcome by changing instructional methods towards a process-based learning environment. Process-based learning environments, where all students work on open-ended problems and are encouraged to investigate and to discover mathematics, proved to lessen sex differences in mathematical achievement (see also Boaler, 2009).

Classroom mathematics word problems may have another facet that hinders students' development. In the field of learning fractions, Mack (1990) has revealed that the sequence of tasks does not correspond to the sequence how students' prior knowledge would help understanding fractions. Concretely, six grade student have ample prior experience about fractions, and they often use partitioning (i.e., dividing quantities into pieces), and thus they can relatively easily understand improper fractions (i.e., when the numerator is greater than the denominator). However, tasks containing improper fractions are usually left to the end of the fraction chapters in the textbooks.

A similar problem has been found with multiplication by Lampert (1986). She emphasizes that in students' mind multiplication is more complex than repeated addition. If we limit though instruction one's mental model about multiplication to additive compositions, the student may fail later in understanding multiplications to continuous quantities. Lampert's and Mack's research results nicely support more general recent principles of mathematics education like the RME mathematization concept. Schoenfeld's (1988) heretic standpoint about the disaster of well taught lessons tells the same story: carefully performed sequence of steps in constructing mathematics gives the message to students that it is the (mathematical) accuracy that counts when doing mathematics. How students' experiences can provide unexpected results in mathematical word problems were documented in research on child street vendors (Carraher, Carraher & Schliemann, 1985; Saxe, 1988). Although from a mathematical aspect larger natural numbers are more difficult to add and subtract, children having experiences with the inflated Brazilian currency were better in adding numbers that could be matched with real prices even if these numbers were relatively large.

Classroom word problems were categorized in several investigations according to features that are both mathematical and of cognitive representation nature. As far as additive structures are concerned, the following types of simple word problems were identified: combine, compare, change and equalize problems (see Radatz, 1983; Riley & Greeno, 1998; Jitendra, Griffin, Deatline-Buchman & Sczesniak, 2007; Morales, Shute & Pellegrino, 1985).

Independently of the task content, students strive for categorizing word problems, and driven by their beliefs about the solvability of word problems, form different strategies to cope with different types of problems. This tendency to categorize problems is not per se a problem, since recognizing the common structure of superficially varying tasks is an important characteristic of true expertise in a given domain (see e.g., Sternberg & Frensch, 1992). However, when finding the operation to be computed and the data to be matched with that operation are generally sufficient for solving a task, it may create blind alleys for students in their mathematical development. Verschaffel, Greer and De Corte (2000) analyze this so-called superficial schema of word problem solving, comparing it to the schema of genuine mathematical modeling. The crucial point is whether the student builds a situation model by means of deep understanding of the problem situation, or (s)he skips building such a situation model and jumps immediately to a mathematical model deemed to be appropriate – based on superficial task characteristics. Illustrating and documenting those blind alleys in word problem solving the reader should consult Verschaffel, Greer and De Corte (2000). A Hungarian study brought further evidence about the presence and strength of superficial word problem solving strategies (Csíkos, 2003).

One important aspect of using word problems in classrooms is teachers' beliefs and attitudes towards realistic word problems. "The teachers seem to believe that the activation of realistic context-based considerations should *not* be stimulated but rather discouraged in elementary school mathematics" (Gravemeijer, 1997, p. 391. – italics in original text). Verschaffel, De Corte and Borghart (1997) empirically documented pre-service teachers' disposition towards giving non-realistic reactions to simple arithmetic word problems themselves as well as their tendency to give higher marks to non-realistic than to realistic interpretations and solutions of word problems by students.

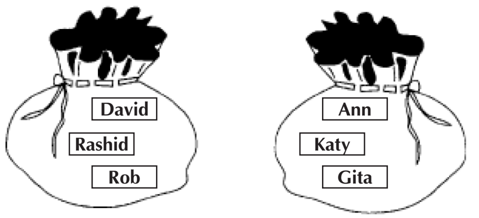
Sociomathematical Norms, Contextual and Content Effects

The term "sociomathematical norms" was introduced by Yackel and Cobb (1996). These norms, which are (in contrast to the broader social norms) by definition restricted to the curricular domain of mathematics, are derived from individual and group mathematical activities (classroom practices). Classroom teachers as representatives of the mathematical community (Yackel and Cobb's experiment was carried out in second grade classrooms) have a crucial role in establishing norms about mathematics and its teaching and learning like what an appropriate mathematical problem is, what an appropriate response to a mathematical task is, how the acceptable forms of explanation and argumentation look like, etc. These norms can vary from classroom to classroom, but "sociomathematical norms are established in all classrooms regardless of instructional tradition" (p. 462).

One important aspect of sociomathematical norms is whether acceptable mathematical explanations in a classroom are mathematical or status-based. Many children tend to infer that their answer is incorrect as soon as the teacher questions it. This norm can easily lead to rigid and false beliefs about the nature of mathematical problem solving and argumentation. Although the analysis of children's mathematical beliefs is beyond the scope of this chapter, it is students' mathematical beliefs that take their share in explaining difficulties in the application of their mathematical knowledge in different contexts and settings (e.g. in mathematics in streets versus in schools, see Carraher et al., 1985). One strong belief revealed in several studies is that a mathematical task always has (only) one right solution, and there is (only) one right way to find that solution (see for example Reusser & Stebler, 1997; Verschaffel, Greer & De Corte, 2000; Wyndhamn & Säljö, 1997).

How sociomathematical norms in general and norms about the role of reality in word problem solving in particular develop can be understood in the light of some theories belonging to sociology and linguistics. Cooper (1994) has successfully used Bernstein's educational knowledge codes, distinguishing between common sense knowledge and school knowledge (also called everyday and esoteric knowledge, respectively). According to Bernstein's argument, children are very early in their school career discouraged from connecting common sense knowledge and school knowledge. Even today it can be revealed that school success depends to some extent on students' willingness and capacity to disclose common sense knowledge as a source of information in mathematics problem solving. Cooper and Dunne (1998) applied both Bernstein's and Bourdieu's insights about the possible social class differences in school (and mathematics) achievement. These differences can be attributed to a relative lack of access to the cultural resources demanded in school situations. Bourdieu's powerful phenomenon of "feel for the game" could be applied in explaining social class differences in some standardized mathematics items. One striking example is the so-called Tennis item depicted in Figure 2.2.

David and Gita's group organize a mixed double tennis competition. They need to pair a boy with a girl. The put the three boys' names into one bag and all the three girls' names into another bag.



Find all the possible ways that boys and girls can be paired.
Write the pairs below*. One pair is already shown.*

Rob and Katy
...

Figure 2.2 The Tennis item. Source: Cooper and Dunne, 1998, p. 132.

Detailed analyses of students' achievements and interview transcripts have shown how the "feel for the game" phenomenon explains social class differences. For esoteric mathematical reasoning, it is clear that children's names and supposed nationality is not a relevant consideration to be taken account of. About one quarter of students aged 10–11 years produced only three pairs instead of the mathematically correct nine ones. However, these children produced three "realistic" pairs in a sense that the three pairs were distinct; each name was used only once. According to Cooper and Dunne, this type of tasks used in evaluation settings raises problems of equity, i.e. equal opportunities in education. How in general mathematics word problems generate inequities (in terms of gender, social class, etc.) is analyzed and criticized also by Boaler (2009).

According to other empirical results, in grade 3, word problems of the story problem type (i.e., where figures and relations are embedded in a narrative story) are challenging for students (Jitendra, Griffin, Deatline-Buchman & Sczesniak, 2007). Nevertheless, in grade 3, word problem solving is a useful indicator of general mathematical proficiency (Jitendra, Sczesniak & Deatline-Buchman, 2005)

The role of culture in mathematics achievement incorporates the role of language competence. To understand mathematical word problems one has to be capable semantically analyze the linguistic components of a task, and furthermore, to identify important and redundant parts. Elbers and de Haan (2005) studied multicultural classrooms in which language components of mathematical word problems are of more peculiar importance. They found that language problems in understanding texts were not solved by means of referring to the everyday meaning of words, but conversations (and students' help-seeking behavior) focused on the special meaning of terms they have in the context of a mathematical lesson. The priority of understanding word problem text genre and context over pure semantic understanding of text cues have been further supported by Morales, Shute and Pellegrino (1985) whose study revealed no language effect on either solution accuracy or on the ability of categorizing math word problems – their subjects were Mexican-American. Nevertheless, well-documented results prove that the linguistic features of a word problem influence to certain extent the solution process (e.g. the term 'of these' may influence whether an appropriate mental representation is built, see Kintsch, 1985).

Two effective strategies to promote connections between students' mental representations and learning objectives to meet can be: rewording the word problem, or personalizing it. In an investigation by Davis-Dorsey, Ross and Morrison (1991) it has been revealed that fifth grade students profited from the personalization of the task (i.e., incorporating personal information about the learner) and second grade students profited from both personalizing and rewording the content (i.e., making the text more explicit, helping to translate its content into mathematical terms). In this experiment, word problems that could be considered as mathematically identical, did differ in their contextual and content features.

Another – even more radical – possible change in improving classroom environment is the use of reciprocal teaching in mathematics. Magdalene Lampert (1990) adapted the instructional method called reciprocal teaching from reading education (see also van Garderen, 2004). The heart of this method is deliberately altering the roles and responsibilities of the teachers and students in the classroom. She notes that this change requires changes also in tasks that define mathematical lessons. As for defining different contexts in which the application of mathematical knowledge is claimed and expected, we follow Light and Butterworth (1992) who gave a rather broad definition: the context of a task consists of several layers of information related to the task: physical, social and cultural settings. Tasks with the same mathematical structure and with the same content can be solved differently according to changes in the context. However, as Verschaffel, Greer and De Corte (2000) illustrate, the effects of context changes, in case of a special class of word problems context changes, may result in only slightly different levels of student achievement. These context changes involved warning messages at the top the paper and pencil tests or embedding the task in a test that contain puzzle type tasks. These slight changes may suggest that context changes more radical than staying within the paper and pencil methodology may have stronger influence on students' solution patterns.

The content of a task can be defined as taking the definition of context as a starting point. We also borrow the expression 'noun term' from Kintsch and Greeno's (1985) seminal article. There is an assumption widely accepted (or at least used) in the mathematics education community: word problems should fulfill the role of providing a parade-ground for mastering arithmetic skills. According to this tradition, changing the content of a task should not necessarily influence students' achievement; what is more, students are expected to develop transfer skills enabling them to solve tasks with the same deep mathematical structure equally well, independently of the current content elements of he tasks. It should make no odds whether the noun terms of a task originate in the micro-worlds of football or fashion or whether some superficial changes are made in the formulation or the placement of the givens and/or the question.

A Taxonomy of Tasks of Mathematical Literacy

In this section a categorization of mathematical tasks will be proposed. There are many aspects that can be starting points for different categorizations. In international system-level surveys (see e.g., OECD, 1999) there is usually a multidimensional model in which tasks are classified according to mathematical content, thinking processes required, and task format. In the PISA studies (see OECD, 2003) the context of the task appeared as a new dimension. The existence of the context dimension and the four values of this scale can be considered as an expression of an educational policy intention of paying ample attention at the applied side of mathematics and of covering a wide range of topics in assessing mathematics literacy.

When applying two or three dimensions (e.g. mathematical content, context, and competency cluster in PISA 2003) and the concrete values of each dimension, a rectangle or cuboid can be used as a model of which there are several cells representing different types of tasks. Now we provide a category-system for an 'application' dimension of mathematical knowledge. This categorization has its precedents in part in the PISA study contextual dimension, but mainly relies on the horizontal mathematization idea of the RME movement.

Challenges and Difficulties in Developing a Category System for Application Tasks

The logic and basis for this categorization is in line with Erikson's (2008) idea of developmental stages in arithmetical thinking. Different developmental stages can be associated with corresponding behavioral patterns and corresponding mental structures. Starting from a possible hierarchy of mental structures, it is possible to match them with corresponding behavioral patterns observable in appropriate evaluation contexts. In this sense, tasks unambiguously belonging to different categories of tasks requiring different behavioral patterns will make it possible to reveal the test takers' corresponding mental structures. However, with respect to the application dimension of mathematical knowledge, there are problems with matching mental processes and observable behavior. A striking example came from Cooper (1994). The so-called Lift problem (Figure 2.3) have become an often cited example illustrating how different possible solutions to an open-ended question can be analyzed in terms of understanding the task as a realistic or routine task.

This is the sign in a lift at an office block:

> This lift can carry up to
> **14 people**

In the morning rush, 269 people want to go up in this lift.
How many times must it go up?

Fig. 2.3 The Lift problem

In Cooper's (1994) analysis it is clear that the expected right answer (i.e. 269 ÷ 14 rounded up to the nearest whole number can be the result of very different understandings and solution strategies. One possible way is to understand that this task signifies a real problem that has to be solved, but taking account of the test condition, students should not create new variables and should not question some axioms implicitly involved in the task. The other way is to understand that this task signifies a routine school mathematics problem but there is a trap in it. In this second way, one should not divide 269 by 14, because of falling to trap. However, as Cooper suggests, the first type right solution requires some assumptions that are almost never true, e.g. the lift is always full except for the last trip. If someone assumes that a lift that is designed for 14 people works on average carrying about 10 persons, will give a wrong answer if only she realizes that in a test one is not expected to create new variables, but to find out the intentions and use the rules such tasks usually require and activate.

There are some classifications of realistic (and non-realistic) mathematics word problems proposed in the literature. One relevant aspect is whether the task classification has a mental representational and instructional focus or whether it has a system-level assessment purpose. The first aspect is representative of a taxonomy proposed by Galbraith and Stillman (2001). According to Verschaffel (2006), this categorization focuses on student thinking processes expected to elicit and on the relationship between word problems and the real world. In this taxonomy, there are four word problem categories:

(1) injudicious problems, wherein realistic constraints are seriously violated;
(2) context-separable problems, wherein the context plays no real role in the solution and can be stripped away to expose a purely mathematical question;
(3) standard application problems, where the necessary mathematics is context-related and the situation is realistic, but where the procedure is (still) rather standard;
(4) genuine modeling problems, in which no mathematics as such appears in the problem statement, and where the demarcation and formulation of the problem, in mathematical terms, must be (at least partly) supplied by the modeler.

This taxonomy focuses on students' thinking (modeling) processes, i.e. how links between their mental representations and the real-world objects are realized.

Another categorization that can also be considered as an important antecedent of the categories proposed in the forthcoming parts of this chapter, was described by Palm (2008, 2009). Palm focuses on task characteristics of word problems that emulate out-of-school situations. He attempts to describe what characteristics a so-called authentic task should have. The key idea is a reference to the elements of 'simulation', i.e. the concordance between word problems and out-of-school, real-world task situations: comprehensiveness, fidelity and representativeness. These terms are borrowed from a seminal work written by Fitzpatrick and Morrison (1971), whose work was made of a system-level evaluation purpose.

Palm's approach for categorizing authentic tasks yielded support from an analysis of Finnish and Swedish national assessment tasks. Although this task battery was made for upper secondary school students, there are some lessons worth considering for lower grades as well. It has been revealed that 50% of the word problems used in national assessment both described an event that might occur out of school context and included a question that might be 'realistically' posed in that event. These two superficial task characteristics may strongly indicate that the word problem is authentic, and authenticity – as described in other taxonomies – is associated with students' genuine mathematical modeling processes.

Our attempt to set up a taxonomy for word problems from the aspect of applied mathematical knowledge will necessary take account of both characteristics of word problems and the mental processes that are elicited in the word problem solving process. There will be four task categories proposed in a way that it may be considered a two by two system. There are two categories for word problems not requiring genuine mathematical modeling of the problem situation, and there are two categories called realistic and authentic that refer to genuine mathematical modeling in the sense of the following description: In accordance with Galbraith and Stillman (2001), genuine modeling problems are problems wherein there is at least one modeling complexity involved that makes that the solver cannot straightforwardly formulate, understand, mathematically represent, solve, interpret, answer the problem in the same way as he can do for a prototype or pseudo-real problem.

"Bare Tasks" Containing Purely Mathematical Symbols

The term "bare tasks" is borrowed from Berends and van Lieshout's (2009) taxonomy for word problems in relation with whether they contain drawings as essential or irrelevant part of the task. Bare tasks contain purely mathematical symbols and at most a formal instruction about what to do or how to solve the task (e.g., "$10 + 26 = ?$"). This category stands here as a sufficient and necessary starting point to define what types of tasks have little to do with the application of mathematics. Tasks containing purely mathematical symbols – or text at most 'solve the equation' type instructions – do not usually have relations with students' applied problem solving or mathematical modeling. Please note, however, that even bare tasks are appropriate means for facilitating mathematical modeling in a way that is called a reverse way of word problem solving, i. e. when students are taught how to pose word problems given the mathematical structure of the task in purely symbols.

This type of tasks is usually part of everyday classroom practice, and the capability to solve such tasks is part of the curricular objectives as well. A possible sharp distinction between these 'bare tasks' and tasks of the other three categories can be found in understanding and learning fractions (Mack, 1990).

We do not want to give the impression that bare tasks are per se easier than tasks embedded in a context. To the contrary, in some cases, children will perform better on word problems than on mathematically isomorphic bare tasks. This has been stressed and documented by several authors (Carpenter, Moser, & Bebout, 1988; De Corte & Verschaffel, 1981).

Prototype and Pseudo-Real Word Problems

As we have discussed in a previous section, classroom instruction frequently uses and relies on so-called prototype examples. These tasks are word problems dressed on a skeleton that can be considered as a representative of a mathematical operation or other mathematizing process. Prototype examples are often called in Hungary 'green stove' or 'precept' examples from which one can induce and explore analogies. We define prototype examples as mathematical word problems that are used in order to learn to recognize and practice a particular mathematical operation (e.g. multiplication) or a particular mathematical formula or solution schema (e.g. the "rule of three"), In such problems, the content is carefully selected or constructed because of its familiar and prototypical nature, but that content has no special meaning or role from a realistic point of view.

Certainly, learning form worked-out prototype examples can be a powerful tool in improving students' mathematical abilities, but there is a potential danger in generating so-called rational errors (Ben-Zeev, 1995) in a way that instead of transferring the deep structure and the solution processes adequate for the prototype example students may rely on surface similarities. (E.g., poor learners may categorize word problems according to their content or contextual features like 'age difference tasks', 'flag coloring tasks' and so on even though mathematically speaking they have little or nothing in common.)

The understanding and solving of many word problems depends on "tacitly agreed rules of interpretation and on multiple assumptions of prototypicality" (Greer, 1997, p. 297.) According to Hong (1995), good problem solver sixth grade students are able to categorize word problems in the early phase of problem solving, i.e. already during the initial reading of the problem. Jonassen (2003) provided an extensive review of literature about students' (mis)categorizing word problems. The essence of these studies, as it can be plausibly hypothesized, is that successful problem solvers categorize word problems according to their (mathematical) structural characteristics, while poor achievers tend to rely on surface (or situational) features (see Jonassen, 2003; Verschaffel, De Corte & Lasure, 1994). It is not mainly the content of the task that elicits such superficial strategies, but the feedback received from the teacher (and from other participants of the school system) about the sufficiency of using such strategies. Many teachers even explicitly teach four- or five phase strategies by which most of the word problems can be successfully solved (e.g., gathering the relevant data, naming the necessary operation, executing the operation, underlining the solution) Teaching such strategies is saluted only if the meaningfulness (or mindfulness) and the flexibility (or adaptivity) of these strategies can be maintained.

Realistic Word Problems

The assessment of student achievement on realistic word problems must, however, be done more flexibly and more dynamically than in traditional former ways (Streefland & van den Heuvel-Panhuizen, 1999).

The term 'realistic' is used according to the Dutch RME definition. In a realistic problem, students are expected (and many times required) to use their mental representations and models in order to understand and solve the problem. Please note that the term realistic refers to mental imageries that are the various means for appropriate problem representations. However, activating and using mental imageries do not necessarily imply that a task is realistic. In Cobb's (1995) understanding, adding two two-digit numbers will not require students to use situation-specific imageries, albeit they probably use imageries during the addition process. Making distinction between realistic and pseudo-realistic word problems the term' situation specific imagery can be of our help.

How to distinguish realistic word problems from the prototype- or pseudo-realistic ones? We agree with Hiebert et al. (1996) that no task in itself can be routine or problematic. A task becomes problematic to the extent and by means of treating them problematic. Likewise, a word problem becomes realistic to the extent it enables students to use their mental images based on real-world experiences. Inoue (2008) suggests helping students validate problem solving in terms of their everyday experiences. It can be done by incorporating fewer contextual constraints in order to let students create a richer opportunity for imaginary construction of the problem. This is in line with Reusser's (1988) observation, who found the various textual and contextual cues too helpful in anticipating the problem solving process. For example, students too often think they are on the right way if the solution process works out evenly (e.g., a division can be executed without a remainder).

In many cases, realistic word problems usually have relatively longer texts than prototype or pseudo-realistic problems do. This is justified by Larsen and Zandieh (2008) in the case of algebra items, where they found it necessary to have a wordy explanation of the situation – when the item is situated in a realistic context. Consequently, the length of the problem text in itself is not a criterion.

A general criterion of a word problem being realistic will involve the following criterion: In a given age-group, for the *majority of students, solution requires mental processes involving horizontal mathematization and genuine modeling elements that go beyond the mere application of a previously taught and well-learnt operation, solution scheme or method. Realistic word problems enable student to build different mental models of a problem situation.* These models may range from mental number lines to a sketched drawing of a rectangular.

Let us illustrate the functioning of this criterion with a task posed by Gravemeijer (1997):

> *Marco asks his mother if his friend Pim may stay for dinner. His mother agrees, but this means that there is one cheeseburger short. There are five cheeseburgers, and including Pim there are six people now.*
>
> *How would you divide five cheeseburgers between six people?*

As Gravemeijer notes, in a real life situation, there can be different practical solutions given: e.g., Marco shares his cheeseburger with his friend, father and mother share their cheeseburger to help out or someone goes out to buy an extra one. Of course, in the mathematical classroom, where all theories of tasks contexts born in the previous decades tell their own story (“feel for the game”, sociomathematical norms, mathematical beliefs, dual educational codes), hardly anyone will propose a solution similar to the above mentioned three renegade answer except for those who do not feel themselves competent enough in division-like tasks. We may hypothesize that more first and second grade children will give renegade, contextual answers taken account of the situation variables than older children would. As for an upper estimation, hopefully the majority of seventh and eighth grade students is able to compute 5/6 as a result of a division called forth by the text of the problem, and without mobilizing situation-dependent imageries. Consequently, this ‘Cheeseburger item’ might serve as a realistic task in grades 3 to 6, requiring students to activate situation-dependent imageries, and find an appropriate mathematical model for the solution. Furthermore, for older children, the task may appear as a prototypical word problem, since they are able to divide 5 by 6, whatever concrete objects are mentioned in the problem statement.

There are useful considerations proposed in the literature about how a word problem may become realistic. According to Boaler (1994), students often do not see the connections between mathematical situations presented in different contexts, and this is because of the (pseudo-real) contexts used in mathematical classroom. She suggests careful selection and construction of word problems in order to develop transferable knowledge from the classroom the ‘real world’. Mere replication of real life situations in word problems is not appropriate. To clarify the difference between word problems that facilitate students’ knowledge transfer from their real world experiences, the following example may be helpful.

De Lange (1993, p. 151.) cited an example from the Illinois State test:

Kathy has bought 40 c[1] *worth of nuts. June has bought 8 ounces*[2] *of nuts. Which girl bought the most nuts?*

a June
b They both bought the same amount
c Kathy bought twice as much
d Kathy bought one ounce more
e You can’t know

1 c stands for cents, i.e., 40 c equals .4 USD.
2 8 ounces is a half pound, i.e. about 22.7 dkg

According to de Lange, the attempt is „admirable”, since solving this problem requires the student to make an appropriate mental model for the situation, and any attempt to use a general strategy like „search for the data, choose the right operation, and execute the computation” would fail. The expected right solution here is “you can’t know”, since the numerical data will not imply any straightforward computational answer. However, de Lange suggests to further improve the task in a way that all options might be true, and it is the students who have to create different task conditions in which the options become true. Furthermore, it follows that the task format in itself can make a problems situation realistic: often it is the open-endedness of a task that makes a given word problem realistic.

In Treffers’ example (1993) the use of newspaper excerpts revealed how children can try to solve without bias a mathematical word problem. Fourth grade children receiving the text saying that “On average I work 220 hours per week” was questioned whether it was possible to work 220 hours per week. Children not immediately mathematized the problem, and give answers of various types. One important aspect of realistic mathematics tasks is to encourage diversity by means of open-endedness.

Contrary to previous assumptions, as Inoue (2008) warns, the benefit of use of familiar situations is limited. What is more, the familiarity of the context seems to be correlated with both the content area within mathematics and with the required level of thinking processes (Sáenz, 2009). For example, open-endedness in question format is more frequently related to higher level thinking skills. – Hence the three dimensions of the mathematical objectives (disciplinary content, applied mathematical knowledge, mathematical thinking abilities) are intertwined, enabling us to consider the application dimension as albeit relatively distinct, but embedded in different category values of the other evaluation dimensions.

Authentic Word Problems

A fourth type of word problems is labeled as authentic. Although it should be clear that the terms realistic and authentic are closely related, we feel the need to use the term authentic word problems to give a specific qualification to a particular subset of realistic word problems. The term ‘authentic’ has been used in various contexts in the mathematics problems solving literature. Accepting Palm's definition, authenticity has several degrees, and it expresses a relation between school tasks and real life situations. When "a school task …well emulates a real life task situation" (Palm, 2008, p. 40) that task may be called an authentic one. On the other way, Kramarski, Mevarech and Arami (2002) approached authenticity from a problem solving perspective. They call a mathematical task authentic if the solution method is not known in advance or there are no ready-made algorithms. A third proposal for a definition comes from Garcia, Sanchez and Escudero (2007) who speak about authentic activities, i.e. the process of relating a task and a real situation.

In itself no task can be considered either authentic or non-authentic (similarly to the lack of distinction in case of the realistic versus non-realistic dichotomy), so when aiming at providing useful categories for an evaluation framework, these three definitions are not equally applicable. As for the first definition, emulating a real life task situation may refer to two things when making decisions about the level of authenticity. First, the degree of emulation may depend on a textual elaboration or creating an appropriate task context (e.g. playing the situation). Secondly, there can be remarkable differences among students in that to what extent a situation can be of familiar (therefore real life) nature. The second definition has even more obviously addressed inter-individual differences (i.e. a solution method is not known for whom?). The third approach is closer to the RME interpretation of horizontal mathematization. In sum, from educational evaluation purposes, we suggest using Palm's definition with emphasis on the need for extensive verbal elaboration in order to "emulate" real life situations.

From an educational evaluation aspect, characteristics of and requirement for authentic tasks can be summarized along two lines. First, authenticity should usually require an alienation from the traditional individual paper and pencil methodology towards more authentic settings such as group working on tasks consisting of various sources of information. Second, authentic tasks in traditional paper and pencil format will be lengthier in text, since descriptions of intransparent problem spaces will result in longer sentences providing cues for missing information and providing also redundant details emulating real life situations in that way. Furthermore, many authentic task will contain photos, tables, graphs, cartoons etc. What is more, authenticity refers to a kind of task-solving behavior and student activity.

It is worth bearing in mind that reaching authenticity as reflection or emulation of real world events and situations is rather a utopia, since the context of schooling and the context of the real world are fundamentally different (Depaepe, De Corte & Verschaffel, 2009). The so-called realistic and authentic tasks do not always measure mathematical knowledge and its relations to real life situations, but they measure the 'feel for the game' as analyzed in the "Sociomathematical norms…" section. Although the 'feel for the game' is a valuable aspect of one's achievement, the possibility of totally different mental representations resulting in the same (right) answer to a task intended to measure the application of mathematical knowledge in an everyday context, urged Cooper (1994) to warn politicians and researchers in a way that

> *Mathematics Education "the English experience [in evaluating mathematical knowledge in everyday context] so far suggest that both much longer times scales to allow for the lessons of research and experience play a greater role, and less political interference in the development of tests, will be needed" (p. 163.)*

As Hiebert et al. (1996, p. 10) suggested, "problematizing depends more on the student and the culture of the classroom than on the task." A problem that can be a routine task in one classroom can be problematic and require 'reflective inquiry' while „given a different culture, even large-scale real-life situations can be drained of their problematic possibilities. *Tasks are inherently neither problematic nor routine*. (p. 10. – italicized by us).

In sum, authentic tasks usually have the following characteristics:

(1) detailed (often lengthy) description of a problem situation emulating real world events
(2) the solution requires genuine mathematical modeling of the situation
(3) the solution process often requires so-called 'authentic activity', e.g. gathering further data by means of various methods (measuring, estimating, discussing prior knowledge about a topic)
(4) in many cases students are encouraged to pose problems and ask questions based on both the given word problem and on their real-world experiences.

Summary

Even though bare arithmetic tasks and prototypical word problems still deserve a place in elementary school mathematics teaching and assessment, they need to be complemented more than was the case hitherto with other, more realistic and more authentic types of tasks, which have recently shown to be more promising vehicles for realizing the "application function" of word problems, i.e. to offer practice for the quantitative situations of everyday life in which mathematics learners will need what they have learned in their mathematics lessons.

By their very nature, those realistic and authentic problems have a greater potential of providing learning experiences wherein learners are stimulated to jointly use their mathematical knowledge and their knowledge from other curricular domains such as (social) sciences and from the real world, to build meaningful situational and mathematical models and come to senseful solutions. At the same time, these more authentic and realistic problems yield – because of their essentially non-routine, challenging and open nature, ample opportunities for the development of problem solving strategies (heuristics) and metacognitive skills that may – if accompanied with appropriate instructional interventions aimed at decontextualisation and generalisation – transfer to other curricular and out-of-school domains. And they involve many possibilities to contribute at the deconstruction of several inappropriate beliefs about and attitudes towards mathematics and its relation to the real world.

An important but difficult issue for assessment is how to make it clear to the learners what is expected – in terms of the required level of realism and precision – from them in a concrete assessment setting. In principle, the question about the mathematical model's degree of abstraction and precision should be regarded as a part of what we want students to learn to make deliberate judgments about, as one crucial aspect of a disposition towards realistic mathematical modelling and applied problem solving.

Within the context of a regular mathematics class, wherein discussion and collaboration is allowed and even stimulated, the degree of precision, the reasonableness of plausible assumptions, and so on, may be negotiated (Verschaffel, 2002). But such unclarities and difficulties with respect to the level of realism and precision are more serious, we believe, when problems are presented in a context that precludes discussion, especially an individual written test, as has been shown above when discussing the work of Cooper, 1994; Cooper & Dunne, 1998). So, if we want to include more realistic and authentic problems in our assessments, as pleaded above, we will also need to pay attention at how we will make it clear to the learner – explicitly or implicitly – what "the rules of the game" are for a given assessment problem.

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3

Scientific and Curriculum Aspects of Teaching and Assessing Mathematics

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Historical Characteristics and Relations of Mathematical Science and Mathematics Education

In this chapter the content of mathematics to be taught in the early period of schooling and the points of evaluation of mathematical knowledge are discussed from the perspective of mathematical science. It is rather difficult to answer the question what mathematics is. Mathematics occupies a special place in the family of sciences and school subjects. The complexity of the question is shown by the fact that this is a scientific problem still studied in a field of philosophy, called mathematical philosophy (see Ruzsa & Urbán, 1966, Rényi, 1973, Hersh, 1997, Gardner, 1998). Before we intend to answer the question in a way easy to understand we exclude several "misbelieves", and views being misleading but widely accepted. First of all, mathematics is not "arithmetic", what is more, is not "the science of quantities and space" as considered long time ago. For centuries, the subject of mathematics has been much wider than that.

Mathematics is often considered as a field of the natural sciences which might have several reasons, but regarding its development, research methods and internal structure, it differs, for example, from biology, physics and chemistry to a large degree (Bagni, 2010). Mathematics is not a natural science since it does not examine the substances, the phenomena, etc. appearing in nature, and its methods are also completely different from those of physics, chemistry and biology, since in these latter ones the basic method of gaining knowledge and checking the knowledge is observation and experimentation. As to its subject and methods, mathematics entirely differs from all other sciences. Mathematics is a science discovering the abstract nature and the interrelations of the structures studied by other sciences and those created by its own internal development, and it arrives at new knowledge by the axiomatic-deductive approach, that is, by using the strict rules of formal (mathematical) logic. Note that some dispute even this definition. What is more, some people even doubt the reality of mathematical objects and formulas outside the human brain.

It is evident that mathematics is a part of human culture. Its results had been used in every historical age, while many important mathematical theories came into being due to problems raised by other disciplines. Earlier it was mainly physics that had great impact on the development of mathematics. Nowadays, it is the enormous progress of information technology that gives mathematical research a stimulus. We are also witnessing the growing demand in social science (economics, sociology, psychology, educational sciences) and also in biology for a strict mathematical foundation. The system of interrelations between mathematics and the other sciences is however much richer than it might be evident from what has been mentioned so far. There are a number of examples that theories, results produced by the inherent development of mathematics had seemed for a long time – perhaps for centuries – completely "useless" outside mathematics, and then they turned out to be exactly what were needed in physics or in information technology. Many people, among them also mathematicians, find relationships between mathematics and arts. For the great majority of mathematicians the different proofs, results and theories have aesthetic value, too, and the smarter, the newer a proof, a result or a theory is and the deeper the arguments and ideas in them are, the higher their aesthetic value is. This aesthetic value is at least as important from the viewpoint of the development of mathematics as its (momentary or perceived) "utility". According to the standpoint of UNESCO, mathematics – as a factor in addition to mother tongue literacy – is the basis of civilization.

Mathematics has a unique position as a school subject, as well, and works of many kinds from literature to the contemporary empirical pedagogical investigations tried to reveal its special status (Mérő, 1992). Looking back to history, mathematical calculations and astronomical observations based on mathematics had tremendous practical importance in various societies (river civilisations; van der Waerden, 1977). This period is considered as the birth of mathematical science – and of other sciences – when mathematics (together with all other fields of sciences which had been differentiated since that time) was closely attached to philosophy.

Initially, teaching mathematics was intertwined with the study of mathematical science. Among the written finds, the Rhind papyrus of Egypt was with no doubt prepared for the clerical social stratum, while the text of the Anastasia I papyrus underlines the importance of proficiency in arithmetic. Ancient mathematics education provided the highest possible scientific knowledge of that time – for the few who could have access to it at all.

In the European culture, after the prosperity of mathematics in ancient Greece, the development of mathematics was hallmarked by the results of Arab mathematicians, later by the scholar monks of the monasteries, and in the period of the Renaissance by mathematicians bringing the rebirth of sciences (Sain, 1986). In the schools of the monasteries mathematical education was built around two fields of the seven free arts, arithmetic and geometry (belonging not to the "trivial" part but to the quadrivium). Both fields were needed for practical reasons, like e.g. the efficiency of work or astronomical problems. From the age of the Renaissance on, the books published met the practical needs of the strengthening bourgeoisie and presented the application of mathematical results by means of examples from trading and from real life, in general, see e.g. Treviso arithmetic (Verschaffel, Greer, & de Corte, 2000).

From the 16th century on, mathematics had got an important role in all curricula which were introduced in the frame of unifying processes of the education (Szebenyi, 1997). According to Smolarski (2002), the Jesuit world curriculum, the *Ratio Studiorum* born in 1599 had not only devoted a significant role to mathematics education, but the didactic guidelines prepared for teachers contained proposals for the use of teaching methods which are still in use even today.

It has become widespread from the 16th–17th centuries that the outstanding mathematicians of the era had regular dialogues at personal meetings and via correspondence, and the quantity and depth of mathematical knowledge was growing so rapidly that the cutting edge research and the school curriculum became separated from each other. Until today, however, it has remained an important aspect in determining the school curriculum that it should be scientifically correct and it should prepare for future (eventual) higher level mathematical studies. In the relationship of mathematics as a science and mathematics as a school subject, the thoughts of Dewey (1933), where he weighs the growing number of the fields of sciences and the increasing knowledge of these fields against the – comparatively less changing – learning capacities of children, are especially valid nowadays.

Thus in the framework of the subjects of modern public education, it has become a basic issue for mathematics education to outline a body of knowledge which is correct and coherent from the point of view of mathematical science, and, simultaneously, is in line with the age-group characteristics of learners. These efforts are made difficult by the fact that the basic notions of mathematical science, like e.g. natural number, function, or set were developed in the more mature, self-reflective stage of the development of this science, parallel to the appearance of *meta-mathematics*. These basic concepts which are regarded essential in the school learning from logical point of view necessarily appear in the early period of learning mathematics when the unfolding mind of a child needs direct experiences, and this excludes the possibility of setting up the syllabus by starting with the basic notions as it is usual in mathematics. The virtuosity of mathematics education lies in the fact that, while taking into account the development process of children's cognition, it is also able to ensure the development of mathematical thinking, and the evolving process of notions.

Mathematical Science and the Present Classification of its Areas

Mathematics is categorized as natural science by many registration systems of disciplines and by the educational policies (HunCRIS, Ministry of National Resources 2010), although mathematics is different in almost every aspect – development, methodology, internal principles – from physics, chemistry, biology and geography. Our ancestors had kept better track of this: for example, when the University of Kolozsvár (Cluj) moved from Kolozsvár (Cluj) to Szeged the name of the predecessor of the present Faculty of Science and Informatics was the Faculty of Mathematics and Science. The distinction has been maintained in other countries: for example, the name of one of the faculties of the University of Vienna (Universität Wien) was also Faculty of Mathematics and Science several years ago. (Today there are separate faculties: Faculty of Mathematics, Faculty of Physics, etc.).

Mathematics is not a natural science studying phenomena occurring in reality, since it works with abstract concepts, and it reveals the relations between them allowing drawing only very strict conclusions. Mathematical theories and concepts certainly come from reality, and this is why mathematics can be successfully used in various areas, and frequently, the same mathematical results are applied in very diverse fields.

It is a natural consequence of the development of natural sciences that the outdated, refuted theories together with all the "scientific results" achieved until that time are deleted, declared unscientific, etc. – just remember how the geocentric world view was replaced by the heliocentric one. The theories serve for the explanation of the repeatable experiments and observable phenomena, and a theory, in general, is considered valid if its "explanation is the best".

In contrast to this, it comes from the special theme and methods of mathematics that the mathematical knowledge consists of "ideas" created by human beings, which ideas do not loose their validity through the centuries, therefore they should not be and need not be "thrown away". Although, it is true that in the course of the development of mathematics, some of the earlier concepts and theories need to be improved in preciseness (see e.g. natural numbers, real numbers, and the Euclidean geometry and the theory of sets, respectively), and some topics become more fashionable so to speak in certain periods of time than others. However, every age builds on the mathematical knowledge of the previous ages, and develops it further. This explains why the mathematical knowledge taught in the majority of primary and secondary schools of the world – at least in the core curriculum – was already known in the ancient times. The most important thing is that they still create the basis of the mathematical science. One of the big challenges of teaching mathematics is to find new topics for young children, as well as to teach this fundamental mathematical knowledge by methods which make it possible to bridge the huge gap – at least in some important areas – between the ancient level and the contemporary mathematical knowledge in higher education curricula containing serious mathematics.

It is very important for mathematical science that this cognitive, theoretical perception have an unbroken arc being as high as possible, since an ever growing number of employees have to apply consciously the most up-to-date mathematical achievements. The educational and workplace context of the PISA surveys reflects these requirements (OECD, 2009).

What we want to emphasize is that the development of abstract thinking and the use of abstract thinking are simultaneous processes in teaching mathematics. The everyday work of mathematics teachers is pervaded by this paradox, and they have to give authentic and comprehensible new information to students by dancing between these ropes.

Similarly to the natural sciences, a strong specialization has taken place in the discipline of mathematics since the ancient times until today: new fields were born partly as a result of the internal development of mathematics, partly on "external" influence that is based on the demands of end-users. The biggest reviewing monthly in mathematics, the *Mathematical Reviews* lists annually more than 75 thousand scientific articles in mathematics classified according to their subject. The latest classification of the mathematical topics consists of 47 pages, where the number of the main areas is more than 60 and they are divided into sub-topics in two further steps (MSC 2010).

Table 3.1 shows that the contextual areas indicated in the mathematical evaluation frames integrate naturally into the main areas of mathematical science.

Table 3.1 Main areas of mathematics

Primary content areas	Main areas according to *Mathematics Subject Classification*
Numbers, operations, algebra	11: Number theory 12: Field theory and polynomials (further topics of abstract algebra: 06, 08, 13-22)
Relations, functions	26: Real functions (further topics of analysis and differential equations: 28-49)
Geometry	51: Geometry (further topics of geometry and topology: 52-58)
Combinatorics, probability calculation, statistics	05: Combinatorics 60: Probability theory and stochastic processes 62: Statistics
Methods of mathematical logic	03: Mathematical logic and foundations

The areas with higher serial numbers not mentioned on the right hand side of the table (e.g. 65: Numerical analysis, 68: Computer science) integrate into the topics mentioned on the left hand side through the areas mentioned here. Similarly, the topics of the primary content areas fit well into the classification by the *Mathematical Reviews* of the main disciplinary areas of teaching mathematics. These are the following: 97 Mathematics education; 97E Foundations of mathematics; 97F Arithmetic, number theory; 97G Geometry; 97H Algebra; 97I Analysis; 97K Combinatorics, graph theory, probability theory, statistics.

Thus we can conclude that on the whole the evaluation frames of mathematics correspond to the present research fields of mathematical science. This is one of the reasons for their selection. The other reason is that the development of mathematical thinking can be implemented through these topics as basic material with the help of modern educational methods. We will see later that the counterparts of these contextual elements appear in the *National Core Curriculum*, as well as in the mathematics curriculum of grades 1–6. The system outlined here is in harmony with the historical-cultural traditions, as well as with the frames of the mathematical assessment prepared in Germany – a country being in a similar position by the PISA[1] surveys – as a result of the educational reforms induced by the poor results. The *Bildungstandard* (2005) accepted by the Conference of the German Ministers of Provinces specifies the following content areas in the systemization of requirements demanded by the end of grade 4: numbers and operations; space and form; pattern and structure, quantities and measures; data, frequency and probability. The handling of classical geometry and measurements as separate areas is widely spread in the educational systems all over the world, and this differentiation has been found in the surveys of IEA[2] organization, too, since the beginning.

1 PISA: *Programme for International Student Assessment*, is an international programme under the guidance of OECD in order to assess the students' understanding of texts, knowledge in mathematics and natural sciences.

2 IEA: *International Association for the Evaluation of Educational Achievement*, organization managing the international assessment of students since the 1960s. A TIMSS (*Trends in International Mathematics and Science Studies*) survey has been evaluating the mathematical knowledge with four year regularity since 1995.

Reflection of the Development of Mathematical Discipline in Hungarian Public Education

The Beginning

The history of the Hungarian public education and thus of the elementary level mathematics education goes back to the 18th century. For a long time the mathematics education in the primary schools – like in other countries – was limited to the previously mentioned fundamental arithmetical and geometrical knowledge known as early as the ancient times.

We have to mention as a specific character of the Hungarian mathematics education that mathematical research, contextual development and raising didactical questions always went parallel to each other. At the turn of the 18th–19th centuries Farkas Bolyai already professed such ideas about teaching mathematics which can be accepted even today (Dávid, 1979).

In the following we review the most important stages and intellectual trends of the twentieth century history of the Hungarian mathematics education.

International Movement for the Renewal of School Mathematics at the End of 19th Century

At the end of the 19th century there was no school subject called mathematics, the corresponding subject was called "study of quantities and geometry". At that time international discourse was started for the renewal of mathematics education, what's more an international reform committee was organized by the German mathematician, Felix Klein (ICMI, 1908).

In Hungary the Mathematical and Physical Society, predecessor of the János Bolyai Mathematical Society was established in 1891. The leader of the reform was professor Manó Beke (1862–1946; mathematician, academician, who was a close friend of Felix Klein). He was supported in his work by excellent partners: among others by Gusztáv Rados, professor of the Technical University (1862–1942, mathematician, academician), Sándor Mikola (1871–1945, teacher, physicist), László Rátz (1863–1930, mathematics teacher) (Beke & Mikola, 1909).

Manó Beke wrote several books for the mathematics teaching in primary schools: textbooks, and so-called “guide books” to teachers (Beke, 1900, 1911). In these books tasks taken from real life and making the students thinking also received a role. Dániel Arany on the other hand established a mathematical journal for secondary school students. He stated his purpose in the following: “To give to the teachers and students a book of exercises rich in content.” The first issue of the journal was published on January 1, 1894. This paper was the antecedent of the Középiskolai Matematikai és Fizikai Lapok (KöMaL / Secondary School Mathematical and Physical Papers). What did this reform initiate and what did they manage to achieve? The modernization of the curriculum was the primary aim of mathematicians. They regretted that the mathematical achievements of the past centuries were not mentioned at all in the schools. At the same time they wanted to reform the teaching methods as wells. László Rátz and Sándor Mikola had already worked out earlier the methods and curriculum called “mathematics teaching that makes you work” (Rátz, 1905). They encouraged students to make a lot of measurements and by this they wanted that learning mathematics be interlaced by direct experience. They also emphasized the importance of mental computation and the need to practice estimations.

Among the topics what they primarily regarded the most up-to-date was the teaching of functions which was probably greatly influenced by the reforms introduced by Felix Klein. Perhaps it was due to this fact that the school had so many excellent students, for example, János Neumann, mathematician, the “father of computer”, and Jenő Wigner, the Nobel Prize winner in Physics. The results achieved in teaching mathematics and in educating research mathematicians can be owed to this method and to the excellent teachers (Rapolyi, 2005).

By means of the Középiskolai Fizikai és Matematikai Lapok they tried to train the students of other schools, too. They issued publications and books. The two volume work “*Mathematics workbook”* by László Rátz can still be regarded an excellent course book teaching mathematical topics through problems. All these efforts can be seen as part of the prosperity in the period followed the Compromise of 1867.

Mathematics Education from the '50s of the 20th Century

The students had learnt a lot from the outstanding professors and passed it on to their own students, but the pace of pedagogical renewal was rather slow. In pedagogy the way of thinking cannot be changed by a command word – in this case this is not a figure of speech, but we actually refer to the "making it compulsory" tendency which happened twice in the history of the Hungarian mathematics education, but in both cases with very low efficiency.

In the years after the Second World War, Albert Szent-Györgyi invited Rózsa Péter to write a new mathematics textbook for the secondary schools. This series was the famous Péter & Gallai textbook (Péter & Gallai, 1949). A new type, mathematically correct textbook was prepared for grade 1 of the secondary schools building on practical application and explaining in a visual way. The books for grades 3 and 4 were written with co-authors, Endre Hódi and Jenő Tolnai, college professors.

The structure and methodology of the book series was pioneer in teaching heuristic thinking and learning mathematics through problems. In line with the educational policy of that age the textbook series of Péter & Gallai was made compulsory in every secondary school. The "introduction" of the new however does not mean that everybody is able to teach according to the new principles immediately. In spite of all the good intentions of the preparatory courses the content change did not everywhere go together with the use of the proposed methods.

Yet it can be said, that during some decades its impact had slowly renewed many fields of the secondary school mathematics education, and especially its methods had greatly influenced the later renewal of the methods and curriculum in primary schools (Szendrei, 2005).

The teaching methods have also been changing step by step. The need for understanding which is of key importance in mathematical thinking has become important.

International Tendencies

After the launch of the first Soviet sputnik the USA expected the improvement of her position in the technical-technological competition from the development of the education system and first of all from the mathematics education and from the education of natural sciences. Thus significant expenditure was spent on these areas. At that time the reform of the school system became important also in other countries, as a result of the recognition of the strategic importance of mathematics and science education. UNESCO was a devoted propagator of the new thoughts. At the UNESCO symposium held in Hungary in 1962 the outstanding mathematics didacticians of the world exchanged views and established work relations for a life-time. Zoltán Dienes (1916, mathematician, researcher) and Tamás Varga (1919–1987, mathematician, researcher) were among them. The stirring lecture of Zoltán Dienes inspired many people to implement their ideas in practice, too.

The group mainly consisting of French mathematicians and taking the pseudonym of *Nicolas Bourbaki* and publishing on this name from the end of the 1930s wanted to integrate the mathematical research by revealing the analogies, parallelisms and other relations between the different fields (Borel, 1998). This work had a great importance in the fact that the mathematicians working in different fields found a common language and that the whole system of mathematics became clearer. From the point of view of the education it is equally important that the mathematics curriculum should not be a haphazard collection of different sub-disciplines (arithmetic, algebra, geometry, trigonometry, analysis), but it should be based on uniform aspects (Varga, 1972, 1988). Inspired by the world tendencies several efforts were initiated in Hungary for the modernization of teaching arithmetic and geometry.

The practice that the different topics of mathematics are covered by different subjects in schools is still valid in many countries. The main reason for this is that in comparison to Hungary less time is devoted in teacher education to studying the subject and the methodology. Even in countries where all mathematical topics are taught in the frame of the same subject it is not always solved that these topics are organically interrelated and they strengthen each other.

Tamás Varga was the only researcher, who wanted to renew the primary school curriculum and methods as a whole. The underlying principle of his experiment was the integrated teaching of mathematics which was put into practice supported by the Hungarian university and college teacher training, since during the training future teachers got wide and profound education in all fields of mathematics. Tamás Varga made an effort for the interweaving of these topics which is called in the international literature as the "OPI project"[3] (Klein, 1987).

The other purpose of the renewal was to turn mathematics from a less favoured subject into a favoured one[4]. The OPI project wanted to destroy the artificial obstacles in front of the mathematical development of the learners. The building up of mathematical notions (for example sets), which were tacitly used in teaching and also the fact that the culture of learning mathematics be available to every student became important. The aim of the project was to achieve that mathematical learning in the secondary school could be based on notions, procedures well-established in primary school.

Effort for an Integrated Mathematics Education

One of the significant movements of the Hungarian mathematics education of the 20th century was the integrated (originally the word 'complex' was used) mathematics education project. The "integratedness" is used in many senses. It means the presentation of mathematics as a whole, that is those who designed and carried out the project were not thinking in terms of teaching arithmetic, geometry, etc. separately. In the name of the project the word integrity also refers to the fact that the idea was characterized by the application of the research results of mathematics didactics, pedagogy, and neurology. Finally, the project was of integrated character because it envisaged not only a methodological reform or a separate change in the classroom curriculum, but also the unity of these two, and tried to make the teaching more attractive and more adjusted to the characteristics of the age-group of the students.

In the 1960s, 70s the implementation of new school concepts – in other fields and subjects, too – had to be approved by the authorities, since only one curriculum and one series of textbooks was in force at that time. Experiments were only allowed under very strict conditions, the success of which had to be proved. In every case those in charge of the projects had to guarantee that the students taking part in the experiment would meet all the require-

3 The project was coordinated by OPI, the National Pedagogical Institute.

4 Presumably significant results were achieved in this field. Recently mathematics belongs to the moderately popular subjects in Hungary, being far ahead of physics and chemistry (see *Csapó*, 2000).

ments that their fellow students participating in the traditional education met. In the case of mathematics this was not very difficult, since the primary school material was very narrow; this was especially true for arithmetic and geometry. As to algebra the curriculum only contained the solution of a simple equation. The negative numbers were only taught in grade eight.

Tamás Varga, the leader of the project, author of textbooks and books popularizing mathematics taught mathematics methodology at the Loránd Eötvös University. At the time of the implementation of the project however he worked at the Mathematics Department of the National Pedagogical Institute. There at the department lead by Andor Cser, later by Endre Hódi he set up a classical "school of teaching mathematics". Those who joined him made tremendous efforts for the improvement of teaching mathematics. He organized seminars, visits of lessons for the university students and he also translated the literature. He disseminated on every forum his mathematical-methodological knowledge obtained by his wide-scale language knowledge. He was in contact with several researchers of the world, extended, controlled, and shaped his concept about mathematics education all the time. He had taken over the good ideas and adapted them to the Hungarian conditions and ignored the false doctrines leading to formalism. Mathematics teachers became co-workers, they had discussed the promising or less good ideas, and either accepted or refused them. The OPI mathematics education project began in 1963 and was continued even after the introduction of the new curriculum.

Mathematics Curriculum of 1978 and its Antecedents

The Ministry of Culture established the so-called Modernization Committee under the leadership of János Szendrei (1925–2011), professor of the Gyula Juhász Teacher Training College. After the visit to the places of the experiment and after studying the written materials the Committee proposed that the OPI mathematics education project be the basis of the new curriculum. The mathematicians provided a lot of help. Many of them, among others Alfréd Rényi (1921–1970), László Kalmár (1905–1976) and Rózsa Péter, academicians, János Surányi (1918–2008) as well as the professional committee of the Hungarian Academy of Sciences provided the necessary technical support.

The school subject was named mathematics already from the first grade. From 1972 on, several topics which had not been covered before were introduced into the curriculum: sets, logic, functions, series, algebra, combinatorics, elements of probability and statistics. Most of them were not part of the education programme of future teachers.

Other topics (e.g. the negative number) were already introduced much earlier than they used to be. Therefore only those teachers were allowed to teach according to the new curriculum, who participated in a preparatory course. The courses were mainly organized by the pedagogical institutes of the counties.

The large-scale introduction of the curriculum was however prevented by the launching of curriculum preparations covering all subjects. From 1978 new curricula were introduced in all fields of public education on phasing-out basis. In the case of mathematics the criteria that only those teachers were allowed to introduce it who would have liked it and had gone through a long preparatory process, was dropped. Beginning from grade 5 a so-called "provisional curriculum" was used in teaching so that the upper grade primary school teachers could also prepare for the teaching of the new topics. New teaching materials and new methods were introduced into the education based on the traditional curriculum of the lower grades. Preparatory courses were organized at central and county levels to the provisional curriculum, too. The *mandatory* introduction of the new curriculum in 1978 was already at that time recognized to be definitely a premature education political decision.

Tamás Varga and his co-workers tried to cope with the problems emerging already in the planning period. During the short courses some participants could not understand the purpose of the introduction of new topics. For example, sometimes they missed the point that gaining knowledge of other number systems and manipulation with them is only a preparation for the deeper understanding of the decimal system. The early use of symbols and names was so attractive that many teachers "enthusiastically" made the children calculate in other number systems. They have made them learn the technical words "set" "relation symbol", etc. without the proper foundation of the notions.

Many of them had shortened the originally proposed long period of time for the development of the number concept, for example that we should give models to measuring with unit by using many different units; the content "number of measurement" of a number should be worked out from the first grade. Some recommended tools were used very formally, etc.

During the change over to teaching according to the new curriculum not only the topics were new, but also the mathematics teaching methods which until that time were only used and proposed by a small group of teachers. There was a need for well prepared, self-educated, creative teachers being able to make independent decisions. For teachers who are experts in the different mathematical topics, who plan and organize the activities of the students in the classroom, ensure gaining experiences by using different senses, work out further steps of abstraction, find out and ensure the diversified use of the means, adjust to the manifestations and comprehension of the child, take into account the age characteristics of the students as much as possible, allow the debate and create a happy and democratic learning atmosphere in the classroom.

It is worth emphasizing again that this was the first curriculum in Hungary, which came up with proposals not only for the curriculum and for the methodology, but also for the ways of the implementation of the collective work of students and teachers, for the creation of a proper atmosphere in the classroom. In terms of current usage it was an educational programme rather than a curriculum.

Tamás Varga considered it very important to show the usability of mathematics in the school where mathematical knowledge gives an efficient help to solve real life problems. He regarded the teaching of combinatorics, probability and statistics and the development of the way of thinking necessary to it as an indispensable mean to the successful application of mathematics. The periods of the curriculum of '78 and the correction curriculum did not bring the required results in these latter areas. In the school teaching of mathematics the applications of mathematics have been presented on a small scale only. The topics of combinatorics, probability and statistics were also pushed to the periphery: most teachers tried to avoid or minimize these parts in education. The work of Tamás Varga made a great impact on the mathematical educational endeavours in the Netherlands, mainly through his contacts with Hans Freudenthal (Freudenthal, 1980a, 1980b).

At the time of the *Second International Mathematics and Science Study* (SIMS) there was an opportunity to compare the results of grade 8 students learning according to the provisional and the old curriculum because 46% and 44% of the representative sample learnt according to the provisional and the old curriculum, respectively. All students solved two series of mathematics problems: a booklet consisting of 40 problems (booklet no. 8), and one of the four booklets, containing 34 problems each (booklets no. 7/A, 7/B, 7/C, 7/D). Students learning according to the provisional curriculum achieved better results than those learning according to the old one not only as to the total scores, but in categories of mathematical knowledge, comprehension and application, and they left less problems unsolved (Radnainé, 1983).

Curricula after 1986

After processing the research results the correction in 1986 had not changed, only modernized the bases and made them more children friendly. In grade four it narrowed the circle of numbers and focused much more on raising the awareness and gaining experience in the field of sets, logic, geometry, combinatorics, probability, statistics, reducing at the same time the requirements in these topics. By means of the development of the area of mathematical thinking it tried to increase the general culture of thinking.

In the 1990s the schools did not actually follow the corrected curriculum of 1986. The catchphrases of democratic publishing and the freedom of teachers made it possible to cut back the diversity of teaching methods. Interestingly, however the use of worksheets, activity books which were regarded so unfamiliar in 1978 became widely accepted. In the education process the preparatory, catalyzing and summarizing role of the teacher was missing in a number of cases.

This situation was made even more difficult by the reform of the system of inspection and its gradual elimination/decay. At the beginning, teachers were happy about the disappearance of the control body, but the lack of an external "co-worker" supporting the practical work, bringing news and new ideas, slowly made teachers insecure. Not only had the controlling inspector disappeared from the pedagogical system, but at the same time the supporting external expert assisting the good teachers was missing, too.

There were several central initiatives for the coordination of the curriculum seemingly going into different directions. Luckily, in the case of mathematics the professional role of the János Bolyai Mathematical Society had strengthened. The problem-centered teaching became more and more popular in mathematics (Burkhardt, 1984; Szendrei, 2007; Kosztolányi, 2006). In summary it can be said that the corrected curriculum of 1986 is the one on which the schools are building up their local curricula even today.

The National Core Curriculum, NCC

The National Core Curriculum introduced in 1995 highlighted out of the topics the foundation of the thinking methods and made it a comprehensive development aspect of all topics. The adjustment to the abstraction capabilities of children and the differentiated development were determined as central tasks. It emphasized the discovery of the relationship between reality and mathematics in the everyday life. It was also recognized that the ability to comprehend mathematical texts needs to be improved. The set-based approach was preserved. The requirements were reduced, but the approach based on activities was highly encouraged.

The topics and requirements were only formulated until the age of 16. The János Bolyai Mathematical Society however formulated a curriculum proposal for the next two years, too. As to the mathematics curriculum in the lower and upper grades of primary schools, its structure and the ranking of the requirements, as well as concerning the methodological recommendations the NCC is a direct continuation of the corrected curriculum. In the lower grades a good basis can be found in the "predecessor" curriculum by the teachers who are outlining the local curricula for the academic year.

In the mathematics requirements of NCC 1995 the applications of mathematics, and probability calculation and statistics were more emphasized than before. As early as the beginning of the eighties Tamás Varga considered the use of calculators and the rapidly developing computers important in education, he himself was looking for the right ways of using them and did not agree with those extreme views which wanted to prohibit the use of calculators, personal computers in the classrooms. We can find several references to the proper use of pocket calculators in the NCC.

The NCC as all other former curricula provoked a lot of disputes. As a result of the criticisms and objections the formulation of the so-called frame curricula started in 1999. They represented a kind of intermediate step, mediating between the NCC and the local curricula, in this way they offered assistance and guidance for teachers.

This had not brought significant change in the mathematics material of grades 1–6; only a smaller shift of emphasis took place as a result of the modification of the formulation. The most important change was the radical reduction of the number of lessons per week, which could question the development of capabilities and skills already expected in the upper grades.

We can conclude, however that these requirements are still valid today when the results achieved in the various fields of the curriculum are to be evaluated on the basis of uniform standards. Also these requirements present the basis for the interpretation of the later NCC versions which intend to regulate mathematics education from the point of view of the development tasks.

NCC 2003 – NCC 2007

The draft of NCC 2002 prepared by the József Eötvös Liberal Pedagogical Society was the antecedent of the mathematical chapter of NCC 2003 (Szendrei, 2002). Its approach was rather different from that of the earlier curricula, because it primarily specified the skills which were set as objectives to be attained by mathematics curriculum rather than the curriculum (teaching material) itself. The development of several skills was also formulated which were until now used by mathematics education, but their development was in a way neglected. Here the curriculum is organically integrated into the developing system of mathematical skills. Since the program was published as an annex to the journal Új Pedagógiai Szemle wide ranges of professional circles responded to it. Many of them highly appreciated the focus on the development tasks against the curriculum-requirement duality of the former NCC. Certainly, many of them expressed their concerns whether those forming the curricula and preparing the teaching materials would make use of the possibilities for which NCC gave authorization. What we can have in mind here is the alteration of the structure of the curriculum, or for example the modification of the traditional solutions of evolving the notion system.

In these documents, an important innovation in the conception of learning of OPI mathematics education project continues, namely that in line with research results it intends to ensure the gaining of knowledge via personal experience. The opportunity for sufficiently wide personal experience has to be ensured for every student – by organizing objective, manual and mental activities – so that under the guidance and/or assistance of teachers they get to the knowledge (knowledge of facts, notions, relations, conceptual systems) by the generalization and abstraction of experiences.

It can be explained partly by lack of conviction, partly by lack of time and in some cases by love of comfort accompanying lack of preparedness that today in many places the main means of education and learning are still board and chalk, exercise-book and pencil, and the teacher's explanation. This happens in spite the fact that in teacher training colleges great efforts are made to pass on up-to-date teaching methods in mathematics to would-be teachers.

It would be of great importance to do research in mathematical methodology to a greater extent in order to support more efficiently the ways in which the process of gaining knowledge and the development of cognitive skills of the age group 6–12 of learners can be assisted by the teachers. Namely this would allow teachers to avoid offering or using methods – for lack of knowledge or for other reasons – which hold back the development of children, prevent or make it impossible to establish their system of notions. The mathematical committee led by Julianna Szendrei developed the material for grades 1–6 of NCC 2003 and of NCC 2007 with the involvement of a wide range of teachers.

Building up Mathematical Topics in the Curriculum for Grades 1–6; Various forms of Mathematical Thinking

Before the curriculum of 1978 the mathematical topics covered basic concepts of arithmetic and geometry and mainly the four basic operations. The most commonly used teaching methods were the presentation, reinforcement and checking. There were great differences among the teachers as to the importance of comprehension, making the exercising interesting and the uniformity of checking.

The new topics included in the curriculum mainly kept in mind the modern structure of mathematics. The evaluation of its legitimacy, importance also divided the professional circles. In the past thirty years, however those topics which characterize the mathematics teaching of young learners were highlighted on international forums, too. And these are completely the same as the topics of the curriculum of '78 (Dossey et al., 2000).

Mathematics has a greater role in the public education in Hungary compared to the international average, and during the preparation of the curricula we draw a lot more from the contents offered by the mathematical discipline. According to the background materials of TIMSS survey (e.g. Mullis et al., 2008) the commitment of the Hungarian public education to teaching mathematics can be recognized in the number of mathematics lessons and in the more accentuated curriculum requirements. This is made possible by our teacher training system which gives a strong foundation in mathematics, and is outstanding compared to other countries as to the number of mathematics classes and the practical training.

In the following we present in detail what problems the teachers had to face when certain topics were inserted into the school material, because this can be an explanatory factor in the analysis of the teaching results.

Numbers, Operations, Algebra

From the point of view of mathematical science, the teaching of numbers, operations and algebra in the early school years is of basic importance. In the case of these mathematical topics a change was proposed by the curriculum of 1978 not only in the content, but also in the teaching methods which are at the same time the corner stones of the successful learning.

One of the specific characteristics of the teaching methodology was that the learners started to become acquainted systematically with the different meanings of numbers as early as from the first grade (e.g. number of pieces, number of measurement, measure of value, symbol). The attempts, as a result of which the content "number of measurement" appeared as an equal partner of "number of pieces" at the beginning of the formulation of the number concept, were received with a still prevailing objection. The virtue of this type of construction is that in this way the notion of a fraction is an organic continuation of the earlier number concept, and it will not appear as a forced additional conceptual content.

The fact that the concept of a negative number and algebra appeared earlier than grade 8 represented a significant change in the content. Great efforts were made to separate the meaning and the notation of a number. (The fact that 2+3 is not an operation, but a notation of a natural number by means of addition gains ground rather slowly). The many different types of notation becoming natural early are the preconditions that a fraction consisting of three elements appear as one number, one object in the thinking of the learner. Or for example the percentage form is also not a new notion; it is only another notation of the number. This systematic teaching method, making learners aware of both the meaning and the notation has today received its theoretical basis in the triple-code theory of Dehaene (2002).

The understanding of the relations between the equality sign and the concept of equality poses similar problems. It is clear from the case study of Ginsburg (1998) that in children's mind the equality sign is a procedure and serves as a notation for a given point of a series of activities ("after the equality sign the solution comes"), rather than the understanding of a case of an equivalence relation.

Every new endeavour provoked huge disputes and objection among teachers, since their standpoint that "we always taught in this way and they still learnt it" was against the new proposals. Teachers began to place great emphasis on making the student understand the meaning of the different operations before automatically learning addition and multiplication. Teachers in general accepted and taught the relations between the different multiplication tables, but often very formally.

An effort can be observed for using uniform symbols in mathematical operations namely that all the four basic operations were introduced in a way that the operand follows the operator (by which the operation is made). In the case of multiplication this effort provoked great objection. For, both the common language and the terminology used in algebra mention the first factor as a multiplier. It is not yet widespread in mathematics education that in the introduction of the operations for 6–7 year old children efforts have to be made for the uniform, comprehensible, clear interpretation. In the case of the already comprehended operation we can use other modifications. The understanding of the psychological characteristics of the creation of concepts has still not become a preferred and mature area for the future teachers.

The "case of division" divides the teachers for other reasons. They hardly accept the idea that in the phase of learning the concept of a *partition* has to be consistently distinguished from *division*. What is more the curriculum of 1978 proposed different symbols for these operations: "/"(slash) became the symbol of partition, while ":" that of division. The differentiation of the two types of divisions was already used in mathematics for medium level teacher training schools a century ago (Pethes, 1901, p. 224). The exact comprehension is essential in the translation of the real content to the operation. Here the fixing of the conceptual difference wanted to be confirmed by the operation symbols. Unfortunately, it is the characteristic of the Hungarian language that no short words exist for the two different types of operations.

The aim of this conceptual foundation is to distinguish partition from division in the developing concepts of small children already at an early stage, since partition provides a basis for the concept of a fraction and helps to understand the division algorithm, and division prepares for the concept of "divisibility", which is a fundamental notion in number theory. The result of a division of natural numbers cannot be a fraction. (E.g.: 7 balloons cannot be divided into 4 equal parts, 7 loaves of bread can be divided equally between 4 families: everybody gets 1 whole loaf and three pieces of a quarter of a loaf). The mixing of the two types of divisions prevents both from developing the concept of a fraction and understanding one of the basic relations of number theory. (Also see the considerations about realistic mathematical modelling, Verschaffel & Csíkos, chapter two of this volume.)

While the Hungarians were envied abroad that the two types of division symbols appeared in the course-books, the welcome was less enthusiastic at home. The problem became a conflict between two professions. The knowledge of teachers about the characteristics of conceptualization in early childhood was not recognized by mathematics teachers advocating the argument that "there is only one division operation in mathematics". They were the ones who often made teachers uncertain referring to their own higher level mathematical knowledge. But this is not about the didactics of mathematics, but about epistemology and psychology.

The non-routine learning of operations encouraging individual calculation methods was also not very popular. About some thirty years later the emergence of the concept of metacognition (see Csíkos, 2007) is luckily in line with the teaching method, which first of all intends to make learners aware of their own calculation method, encourages them to learn calculation patterns from the classmates and the teacher, and prepares the automation of the procedure providing the proper confidence to the student. When using this method the teacher is expected to be more flexible and to accept and follow the "correctness" of several kinds of algorithms.

The importance of estimation was also highlighted. Putting estimation in the foreground did not mean to neglect mental computation but on the contrary to encourage it.

It was also in the curriculum of 1978 that the idea emerged that fractions, negative numbers should be introduced in the lower grades in order to extend the circle of numbers. Although the idea seemed extremely strange (the negative numbers were first dealt with in grade 8 until then), finally it was accepted both by the majority of teachers and the profession. During the elaboration of the details however only a few textbooks include guide-lines for the proper foundation of the "fraction as a number". The development of the concept often gets stuck at the meaning of a fraction as a relation. This gives an explanation for the fact that learners have difficulties with the concept of a fraction. There is a great leap between the relations of "one-third", "quarter" and the indication of a fraction on the number line.

In the teaching of the calculation procedures and calculation algorithms even the cheap pocket calculators and the requirement of the compulsory checking of book-keeping by machines did not put an end to the hegemony of the importance of written algorithms. The importance of understanding an algorithm at the expense of simple swot got into the foreground as a result of serious professional disputes. There are still many advocates of the standpoint that "it is not a problem if he/she does not understand it if he/she is able to do it". The difficulty of the division algorithm lies in the fact that the estimation of the product of a multi-digit number by a one-digit number is not practiced properly, it does not get to skill level. However, this is the basis of the division algorithm. Probably the algorithm should be introduced at an older age than today, when the equivalence of the two types of divisions is already understood by children and they are able to appreciate the algorithm as a human achievement. This algorithm will have significant role later in the secondary school algebra.

It is a general dilemma of public education that from the very beginning of the education of young learners on a balance need to be found between the acquisition of practical knowledge (formulas, "prescriptions") and the cognitive, problem-solving, relation-oriented development of thinking.

The purpose of teaching the elements of number theory is much more than just grouping the numbers into "even and non-even" classes. Paying attention to the properties of numbers makes it possible to examine numbers as entities, in this way making the numbers our "personal acquaintances". The consideration of the characteristics depending on the choice of the number system can be used to strengthen the concept of the decimal system. The topic makes it possible to exercise and extend the basic logical knowledge.

In this process the recognition that the properties of numbers can also be formulated by negation (e.g. number 7 has both attributes "odd" and "non-even") can be a turning point. During the finding of hidden elements, numbers in the game of twenty questions the content of the logical "and" is strengthened. This method also shows the difference between a conjecture and a proof as the difference between random questioning and purposeful exclusion of elements.

Regarding the properties of numbers the simple deductions can also be acquired. By this arithmetic becomes highly suitable for the development of thinking methods of learners.

It was also in the 1978 curriculum that the elements of algebra appeared as teaching material in the lower grades and they have stood the test of time. But what does elementary algebra actually mean? At the beginning this is introduced as a trick like "I thought of a number which is by one less than 9" ($g = 9 - 1$). The "frames" are introduced in connection with the rule games which at the beginning convey the content easier than letters that many different numbers can be substituted into them. (E.g.: Find and write in the frame the numbers between 1 and 20 for which it is true that the number in the frame + 1 > 13. Or even this form must be natural: 13 < number in the frame + 1. This means that the relation between the concept of an operation and a number does not come to an end by "calculating from the left to the right and we get a new number as a result". The operation symbol might refer not only to the process, to the operation to be carried out, but it might represent an object as well.)

The reason why this step is so important is because in the period of the development of the concept of an operation not only the attribute "number as a result of a process" of the number concept develops but also the attribute "number as an object" does simultaneously. The ability to handle this duality jointly is absolutely essential from the point of view of understanding mathematics. The notion of *procept* coined from the words *process* and *concept* is used for the didactic highlighting of this duality (*Gray* and *Tall*, 1994). Thus number is a *procept*, and this is the main reason for the difficulty of creating this concept. The efficiency of teaching depends mainly on the fact whether we are able to make this duality and the switching from one conceptual aspect to the other natural for the students. (In the case of 13 < number in the frame + 1 exactly this mental process is expected from the learners. In order to comprehend the "number in the frame + 1" part and to do calculations the process aspect has to be understood.

To get the result one needs to handle the object aspect of the "number in the frame + 1" that is to compare it to 13.)

A further step forward in learning algebra is the investigation of the truth or of the equivalence of rules describing the "operation" of machines and of tables with rules. Figure 3.1 shows one of these problems.

Select which relation is true for all columns of this table!

♥	4	7	9	5	11	13	15	19
Δ	2	5	7	3	9	11	13	17

a) ♥ : 2 = Δ
b) ♥ – 2 = Δ
c) Δ + 2 = ♥
d) ♥ – 3 < Δ

Figure 3.1 Example of problem of type "machine with rules"

Through these problems the learners gain experience which prepares the topics of identities, transformations of identities and equivalent transformations. The solution of these problems is based on the induction of rules and so it also develops inductive thinking. At the same time these tasks also contribute to the development of the concept of a function, if proper teaching and learning methods and related activities are used.

In connection with word problems the development of the pre-algebra knowledge can be assisted in a natural way: the process of translating the problem into the language of algebra. However we can quite often experience haste and fast solutions forcing formalism. Some of the textbooks for grades 5–6 encourage very early the teaching of formal procedures and the manipulation with terms the understanding of which could so far not happen.

Whereas the separate solution of individual cases, activity (e.g. let the length of the stick be equal to the route taken during the first day. ...), drawing, drawing models and their translation into the language of algebra is a time-consuming process. But before their usage as a general model their deep understanding close to the concrete case is necessary. (From the point of view of the future understanding of algebra and the meaning of formulas it is not the same if in the relation $r + 10 = 100$ the student thinks that r means the eraser, or he/she knows that in this case r represents the price of the eraser in the same currency in which the data 10 and 100 are given in the problem. At first the process will not stop in the formal solution even if the approach is not correct, but later the learner will reach a gap, that will be very difficult to bridge due to the hasty abstraction.)

Relations, Functions, Series

The content area of relations, functions, and series serves both the possibility of future development of "logical thinking" as it is expected by everybody from mathematics and also the development of mathematical concepts, models.

The whole topic was completely new compared to the curricula before 1978. The drawing of "line pattern" which can be regarded a predecessor of the topic of series, was very good, but unfortunately it had worn out from the teachers' repertory. It serves well the introduction of the informal concept of series and its usage as decoration emphasizes the aesthetical character of a pattern, which contributes to the development of a positive attitude towards mathematics by suggesting that 'mathematics is beautiful'.

At the early period relations serve the recognition, and highlighting of certain properties. Their linguistic formulation and notation system represent an entrance to mathematical communication. When children learn to find their ways in certain actual relationships, they discover relationships between things, notions which contribute to their better understanding. The above mentioned activities develop the shaping of basic forms of thinking, the ability to overview the relations.

In grades 1–6 the "model role" is much more underlined (relation, function, and series as a mathematical model of a real problem). An important aim of the topic is to develop the skill of recognition of relationships. Obviously, a lot of elementary knowledge is revealed in connection with the three topics, but the most important is the process of the development of the skill to recognize relationships, and not some symbols or notions which should be memorized.

It is important to develop the thinking in terms of proportions, the negligence of which is still typical in the schools. Perhaps it is at this point where the empathy of teachers is missing the most, as sometimes they do not see that the development of children's thinking is a slow process, they do not understand the need to present a wide range of experiences. They think that it is an easy thing to replace the seeing and feeling of constant growth by the proportionate (multiplied) growth. The fact that in many cases they insist on the conversion of units well before thinking in terms of proportions being properly developed is an indication of this professional failure.

The task "which is more, half an hour or 50 minutes?" can be solved by early proportional thinking. In this case the child thinks 30 + 30 = 60 that is half an hour is 30 minutes and the solution is based on this. If it is recommended him/her to solve the problem by dividing 60 by two, he/she should use the inverse proportionality logic

$$\frac{60}{2} \cdot 2 = 60,$$

which he/she perhaps acquires only at the age of 11–12. Similarly, learners are expected to use inverse proportionality logic at a very early stage (even in the lower grades of the primary school) in many problems of unit conversion formulated with less consideration than necessary.

The practice of teaching relations and functions prepares for using written and drawn models. The number line, the tables, the graphs, the parallel number lines, the rectangular coordinate system as models are integrated into the communication means; in this way they make it possible to achieve the higher level conceptual thinking. By means of them the topic of relations and inverse functions which is fairly elementary at the beginning can be approached at the given school level.

A number of mathematical topics make it possible to develop also the topic of relations and functions. E.g. geometry contributes by studying parallelism and perpendicularity of straight lines as relations, etc.

The wide-spread teaching method of the topic, which is highly suitable for grasping the difference between conjecture and proof, unfortunately, does not lay enough emphasis on the development of the need for proving (Csíkos, 1999). Many teachers are content with stating a rule which appears in a series, but the idea that the relevance of the rule to any natural number is not evident does not turn up. The same applies to the fact that the use of the formulated conjecture needs to be proved. These cognitive processes can happen in case of solving fewer routine problems and also in the constant presence of doubt as natural human characteristic.

Geometry, Measurements

There were a lot of international disputes about teaching geometry in schools. The traditional axiomatic teaching of geometry was an obvious failure. The *Bourbaki group* had struggled a lot for getting rid of the didactic solutions based on visualized settings not only in the mathematical discipline, but also in the school practice. "Euclid must go!" was the slogan due to Dieudonné (Robitaille & Garden, 1989). As a result of this geometry was practically ousted from the subjects taught in many countries. The only remaining topics were the calculation of circumference and area, which were slowly added by the study of some basic forms and patterns in the '80s. The Hungarian mathematics education however did not follow this tendency. There was a change here, too compared to the curriculum before 1978. The change took place not only in the curriculum, but also in the recommended presentation of the teaching material.

In teaching geometry the method of starting from individual cases came into the foreground, which was rather unusual in lower grades before. (That is we do not start from the concepts of point, line, section, etc.) For example in connection with buildings, the spatial forms are considered earlier than the plane ones; a lot of work with quadrilaterals, their grouping according to their characteristics precede the definition of a quadrilateral, etc. In contrast with this, earlier the accent was exactly placed on starting with the discussion of the special cases. The adaptation to the conceptualization of young learners started to gradually pervade the way of the development of geometrical concepts. According to Rickart (1998) one of the specificities of geometry is (compared e.g. to arithmetic or algebra), that the concepts used are very close to the everyday language, thus the majority of learners find basic geometry very easy. The creation of geometric definitions is integrated into the communication means of students slowly, as a result of many years of preparatory work, using their own naive, everyday notions and filling them with precise mathematical content.

One of the models of the different levels of geometrical concepts was developed by *van Hiele*. The explicit impact of the model can be well seen in the American mathematical evaluation frames (NCTM, 2000). The model of van Hiele contains the sequential levels of learning geometry built one on the other (van Hiele, 1986; Senk, 1989). Based on this, learning geometry starts from visualization (that is from attaching visual notions to their names), and finally the geometrical thinking gets to the rigor of deductive analysis. It is remarkable that although this sequence and the building on each other can be observed in the history of science and in the development of children's cognition, the following of this development path often has a subordinated role in teacher education and the deductive setting becomes dominant.

The basic concept of the so-called SOLO-model (*Structure of the Observed Learning Behavior*) is similar to the model of van Hiele, that can be used on the Piagetian and Bruner–Dienes basis for the interpretation and evaluation of the knowledge of mathematics (and in particular of geometry). Compared to the van Hiele levels, a further element is that the cycles of knowledge development are connected to the typical age stages. The term cycle is used intentionally, since the same mathematical principles and concepts are learnt and represented mentally again and again during our life. For the age group of 6–12 we are concerned with the lesson of the SOLO-model is that the written linguistic elements and the mathematical symbols join the sensor-motorized (enactive) and iconic knowledge elements brought from younger age during the schooling years. It is typical that during the learning cycles learners first pay attention to a given aspect or data, then several sources are used and finally they create a coherent view for themselves and then, in the next learning phase a new cycle begins. The education and evaluation stages can be attached to the elements of the learning cycles (Pegg & Tall, 2010).

The activity of learners and the creation of geometrical constructions are serving various developments and contribute to the preparation of several different geometrical areas. This is the starting point of the teaching of geometric transformations, which was also a new thing in the curriculum of 1978. Talking about constructions contributes to the emphasis on properties, and do not "impose" the expressions on children, but they "emerge" as specified linguistic forms searched by children themselves. (For example the collective name for the quadrilaterals cut out of a paper strip can be "trapezoid". Instead of the straws composing the edges of a built tetrahedron we say after a while edge, etc.).

Also the notion of an "angle" can appear as a term helping communication when e.g. children use only the terms wider and narrower in a spontaneous way in connection with a deltoid shape kite. Teaching the notion of an angle has not reached yet the depth expected. In many cases it is not mentioned as a notion which makes it possible to differentiate shapes, situations (e.g. what can the word 'strong' mean in the following technical text: "having a strong slope the efficiency of rain practically ends as the rain stops"). Many textbooks deal only with measuring angles and with teaching the terms in connection with angles. This absolutely does not prepare the concept of trigonometric functions used e.g. in the secondary school. A fewer number of paradoxes enter into teaching perhaps on the grounds that "they do not want to confuse children". Whereas a well-prepared paradox provokes internal motivation, contributes to the imprinting of the acquired knowledge into the long term memory.

The preparation of constructions with given conditions, when learners act according to the instruction "produce as many solutions as possible", especially fosters combinative thinking in an early stage. The use of special construction tools, the modification of conditions, in particular making them stricter, may contribute to the understanding of the Euclidean constructions.

The teaching of the elements of metric geometry was not regarded a new topic. It was however a new thing that the theoretical foundation of measuring already begins in the early school years, and also the principle of approximation is mentioned, what is more, used early. Also the proposed teaching method which calculates circumference, area and surface in concrete special cases, but postpones and extends the period of teaching the "formula" for them is incomprehensible still today.

Several subsections can be defined within the content area of geometry and measurements both for the use of curricula and textbooks and for evaluation. We note here that in the international surveys conducted by IEA, geometry and measuring appeared as two separate fields. This was partly due to the period of transition to SI. The creation of different geometrical shapes (their manipulative, later image level construction) and the transformations of geometrical shapes form further subsections in the area belonging to geometry "in stricter sense". Spatial orientation can be also considered as a subsection, although this and the content requirements of the cultural areas Our Earth-Environment and Man in Nature overlap in many places.

Combinatorics, Probability Calculation, Statistics

The inclusion of these topics into the curriculum caused a really big turmoil in 1978 (C. Neményi, Radnainé & Varga, 1977). On the one hand many teachers had not learnt them in the training institutions. The understanding of pupils' thoughts, and the possible corrections posed challenges for teachers in these topics. This can also be a reason for avoiding these topics. The parents themselves were not familiar with them (besides those studied such topics in higher education), moreover talking in the classrooms about colouring flags or throwing dices seemed too playful and a waste of time. The negative criticisms which were actually attacking the new mathematics curriculum made teachers the most insecure in teaching exactly these topics. We may say that the impact of these facts is still felt today. A longer period of pre-service and service training is needed in order to arrive at the stage when teachers accept an adequately flexible way of thinking together with their pupils. They should realize the importance of developing the learners' thinking and covering the curriculum simultaneously. The teaching of combinatorics and probability can offer a lot of possibilities to do this (see Rényi, 1973), but any approach tending towards formalization, axiomatization, can bring about just the opposite. Empirical research works also called the attention to the necessity of the development of probability thinking in our school system (Csapó, 1994; Bán, 1998).

The probability and combinative thinking are connected by the inductive character of the approach, that is in psychological sense, by the empirical, inductive and in many cases divergent thinking. For, it is just the combinative thinking which is not satisfied with finding only one solution to a system of conditions, but takes into account all the cases. By using this knowledge the optimal solution fitting the given situation can be found. The probability and statistical way of thinking, which is also called stochastic or correlative thinking in the literature is able to think and make decisions in connection with non-deterministic events. What is more scientists and also common people make statements and decisions in connection with deterministic events on the basis of regarding them stochastic (for example in meteorology, in case of timetables, when using simulations, etc.).

Since these ways of thinking have the special character that they say a lot about uncertain events by means of mathematical methods, this type of thinking should be developed in education as early as possible (since they will anyhow develop in a spontaneous way towards some direction in everyday life), see Vancsó (2010). In the opposite case we will see such distortion, "rigidity" which makes it impossible to consider various alternatives, to select from them, or to make rational decisions related to deterministic events. Similarly to other mathematical fields the natural human thinking (and within this the components of thinking already developing in pre-school age) can be regarded mathematically a “misconception”, to put it nicely, it is an incorrect heuristic. Such deviation was discovered in the field of probability and combinative thinking by Tversky and Kahneman (1983), see also Even and Tirosh (2008).

The teaching material is not really big, but the cognitive development is the result of a long process. Not to mention that it is expected that the intuitive thinking related to incidental events should draw on school experience.

The topic of probability can be a good terrain for making operations with fractions, either the probability of the event is given in percentage or in other form. For the time being the teaching of this topic is implemented in classrooms to a lesser extent than it was hoped by the designers of the 1978 curriculum. The operations with fractions are included in the teaching material as before, while the calculation of probabilities as a practical terrain is omitted by the designers of the frame curricula and so omitted from the local curricula.

Omitting elementary calculations from the curriculum has the danger that the concepts of frequency, relative frequency, statistical probability, arithmetical average are not rooted in problem solving; the conceptual preparation of the data representations which can be expected in grades 1–6 will perhaps remain mere formality (verbal definition).

The Content Area “Methods of Mathematical Thinking”

The “methods of mathematical thinking” first became part of the school teaching material with the introduction of the ’78 curriculum. The direct dealing with the topic of sets was included in the majority of reformed international curricula, but the way of discussion and its formal character (use of definitions, symbols) was not at all as low-key as in the case of the Hungarian curriculum. Although the Hungarian teachers were also very soon ready to use more technical terms in order to prove “how serious” mathematics they are teaching. (More details about the psychological aspects of mathematical thinking can be found in the first chapter of the present volume.)

At the beginning, mathematics teachers could hardly accept how important it is to compare persons and objects, to select from and to order a multitude in the “perception of characteristics”, in the development of voluntary observation, in the stabilization of experience and in the formulation of observations.

The matching of the relations of concepts and the relations of sets (for example the concept of sub- and super-ordinate) was a new topic. It was at that time that teachers began to recognize the close relationship between the progress in mathematics and the use of the "linguistic-logical" phrases used in the standard language ("every …", "there is a …" etc.). That is, the statements containing quantifiers and their negations were consciously processed without the use of their names. Among others exactly these linguistic-logical terms may be missing from the language of disadvantageous children when entering school. Therefore the elimination of these deficiencies is the basic pre-condition of mathematical comprehension. Similarly, the development of understanding texts which is necessary for problem solving is of fundamental importance in mathematics classes (Csíkos, 2003).

The appearance of the elements of logic contributed to the development of the practice that students should also formulate statements and not only learn the statements formulated by others as was the case before. The introduction of the "open sentence" topic provoked a great dispute. The situation was not improved by the fact that the topic was already part of the curriculum in other countries. It was difficult to accept that this – the logical function, a wider category also including equation and inequality – is included in the curriculum.

The need for proving should be developed as early as grade 1, but the progress in this case can be very diverse depending on children. Nevertheless, it is worth expecting learners to doubt, reason and prove during the discussion of any topic, and to make an active progress in these areas.

All the efforts to be made in the lower grades in this field of the curriculum were very much attacked by the upper grade primary school teachers. It was a frequently advocated slogan in articles, in the staff meetings of teachers that "you just teach children to count, and we will teach them in the upper grades how to think". This is a misconception not only because the teaching of counting also requires and offers significant cognitive development. This slogan represents the attitude which interprets calculation as a mechanical skill and does not take into account the fact that important strategic changes are taking place in the simplest counting operations, even in adulthood (Lemaire & Lecacheur, 2004). The importance of the development of thinking was expressed by Professor János Surányi in the memorable sentence at the László

Rátz meeting: “I do not know if children can be taught to think, but I do know they can be successfully made to get rid of the habit of thinking.”

One can say that the teaching of this chapter of the curriculum has made a rather slow progress till now, however it became part of mathematics education in Hungary. It was NCC 95 where a separate chapter “*Thinking methods*” appeared for the first time, and the topic “*Sets, logic”* which was indicated in the ’78 curriculum separately was included in it. This at the same time produced a radical shift, because this was the “moment in the curriculum” when the development of thinking was not presented as a kind of by-product of teaching mathematics but as the focus of the teaching process.

The grasping of the essence of mathematical reasoning and the description of its methods is a key, but at the same time a rather demanding issue (Byers, 2007). It is not easy to formulate what mathematical reasoning is, but it is a fact that it differs from everyday thinking. We can often hear in connection with certain arguments or opinions: “one can tell that you are a mathematician”, “this comes only to the mind of a mathematician”, etc. This is most likely due to the fact that mathematics makes people formulate precisely, pay attention to the details, to the importance of weighing of circumstances and information. It accustoms us to recognizing if data are missing, if different things are “merged”. Its two most important pillars are abstraction and obeying the strict deduction rules.

The following factors have an important role in mathematical reasoning:

(1) recognition of the essence,
(2) ability to understand and simplify the complex systems and systems of condition,
(3) combination skill,
(4) making models, abstraction,
(5) intuition, formulation of conjectures (e.g. by studying problems, special cases),
(6) algorithmic thinking (e.g. deduction).

The application of these skills and the active experience of these intellectual activities are necessary even if somebody wants only to use a mathematical method, or wants to order a mathematical investigation. These skills are also needed in error corrections. To a certain extent, these abilities are essential to everybody in order to find his/her way in the everyday life of our modern, technology- and computer-centred society.

Many mathematicians are of the opinion (Dudley, 2010) that one of the most important purposes of teaching mathematics in public education – certainly besides teaching counting – is reasoning, that is the development of the above-listed skills. Not long ago, Dudley's article was published in a journal of the *American Mathematical Society* the title of which – at least in this monthly – was rather astonishing, albeit immediately the first sentence explained the subject: "*What is mathematics education for?*" In this paper the author argues that besides teaching how to count there is no other purpose of teaching mathematics but the development of reasoning. His main argument for this is that higher level mathematics is not used by the employees even in jobs where such knowledge is expected by the employer. Later editions of the monthly (AMS, 2010) published several readers' comments. One of them supports the argumentation of the article saying that according to his experience most of the natural scientists and risk analysers use at the maximum Excel and primary school level mathematics. Other contributors also deal with the level of knowledge demanded from the employees. In the dispute it was also mentioned that mathematics itself is a part of the human culture, just like literature, history, music, etc. which are also included in the school curriculum not because they are needed in later jobs. On the other hand each of those making comments lets it pass unchallenged that mathematics education develops reasoning and that there is a correlation between the level of mathematical knowledge and the capabilities of an employee.

Mathematics education is very important from the point of view of the development of reasoning. Certainly, we do not believe that other school subjects do not develop logical thinking. Our task is exactly to find the specific features that we can offer in the frame of the mathematics subject. The existence of the mathematics subject, however, does not result automatically in the development of logical reasoning. By finding the correct methods the permanent interest can be maintained, and the checking of the received information, as well as the experiencing of the excitement of recognizing the dependence of an effect upon a cause can become a habit (Pólya, 1945, 1954, 1981).

It is also not self-evident, what the possession of knowledge means. In order to achieve that somebody be able to use Excel effectively – that is, not only for entering words into a table, but also for making more complicated calculations, tables, statistics – he/she should first know and be able to use the built-in functions, the understanding of which – with a few exceptions – is beyond the mathematics curriculum of the primary school, and in many cases even beyond that of the secondary school. On the other hand – and this is a more difficult task requiring more serious thinking – having a practical problem, he/she should find out how to translate it into the language of mathematics, and then how to implement it with the available functions. It is not an easy task to teach facts, methods in a comprehensible, not a "parrot-like" way. The main ambition of mathematics education as a whole should be to emphasize learning that makes sense of things from the youngest age.

How Independent are the Mathematics Topics in the Curriculum?

We have to take it into account that the topics of the curriculum formulate not simply teaching objectives but at the same time involve the personal development enhancing the learners' mathematical thinking. Some aspects justify the discussion of the mathematical topics parallel to each other (but not in a mosaic like way). For example the development of the contents "number of pieces" and "number of measurement" of a natural number goes together with the development of the concept of an operation and they strengthen each other. Both can be confirmed by the appearance of the topic of series first as a new problem situation. Parallel to this a number series as an object can provide an efficient starting internal image for the learners to the conceptualization of the set of natural numbers. Covering the topics together can also offer several didactical implications.

Naturally, the inherent logic of the didactic construction of the topics makes it necessary for the learners to devote a longer period of time to each topic. In this way they can perceive the special internal logic, "play field" of a given topic.

The didactic method of merging the topics, which was one of the basic elements of Tamás Varga's didactic concept seems to be implemented in the education practice (Halmos & Varga, 1978). This method makes it possible to present the unity of mathematics, the interrelation of topics according to the development of learners' thinking already from early school age. The small steps of the development of thinking do not make it possible to make a big progress in a topic. The simultaneous development, however, ensures that learners progress in the building of all mathematical topics until the point allowed to them by the development level of thinking. On the other hand it guarantees that leaps in thinking and forced progress at the level of concepts should not be imposed on the learners. The progress in the abstraction level of thinking makes it possible to use the symbols of higher and higher level and to develop reasoning about mathematical objects and to improve the verbal formulations.

The progress learners are making in various topics is related to each other. For example before the development of the appropriate level of thinking in terms of proportions it is totally incidental if students can perform a conversion of units. This can also hinder students in the function type interpretation of relations between geometrical quantities.

The methods of mathematical thinking gained in small steps can provide an impetus to the elements of knowledge. This provides the fabric which integrates the disintegrated cognitions into competencies and makes the growing child able to get to higher and higher level of abstraction, to focus on the essence, and to recognize structural frameworks. Thus after completing grade 6 young learners will be more and more able to use mathematical reasoning.

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4

Diagnostic Assessment Frameworks for Mathematics: Theoretical Background and Practical Issues

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Introduction

The main aim of this chapter is to link the previous three theoretical chapters and the detailed descriptions of content to be presented in the next part. We also intend to deal with the particular features of the frameworks and the reasons why we have chosen and applied certain solutions.

In the first two chapters, the outcomes related to the development of mathematical thinking and generally to the role of mathematics in developing thinking are outlined based on international research and primarily applying the results of developmental psychology. In Chapter 2, the external aims of mathematics education were presented making use of international research results on the application of knowledge. In Chapter 3, the traditions of Hungarian mathematics education and its curricular characteristics were shown as well as the practice the diagnostic system is to be adapted to.

The diagnostic assessment system is elaborated in three main domains parallel with each other according to the same principles[1]. Putting reading, mathematics and science in the same frameworks is supported by several

1 This chapter also contains sections which can be found in the chapter of the same function in the three volumes.

psychological principles as well as practical educational needs. A proper level of reading comprehension is required to have access to mathematics and science and also in turn learning mathematics and science improve the skills of reading comprehension of texts different from literary readings. The logic of mathematics and language can mutually reinforce each other. Science seems to be the most suitable field to apply the relationships acquired in mathematics in practice. In the beginners' stage of education it is particularly important to rely on and to make use of the various systems of relationships, when students' cognitive development is particularly fast and highly sensitive to stimulating effects.

The parallel treatment of the three domains can be beneficial to each other and the ideas and solutions can be very well used in the other field as well. Writing items and developing assessment scales, data analysis and systems of feedback make it necessary to treat the three fields parallel according to certain shared principles. However compromises are to be accepted, as the same principles can be applied in the same way only to a certain degree in the three fields. In order to be consistent, the three-dimensional approach is preserved and applied. Still the peculiarities of the fields will be taken into consideration in the description of various dimensions.

Another benefit of the work carried out parallel is the complementary effect. The theoretical backgrounds to the three domains are presented in nine chapters. In outlining the structure of the various chapters, we were not keen on the strict parallels thus it was possible to set forth one issue in one particular domain or to highlight another issue in another field. By way of illustration in the volume on reading in Chapter 1, the aspects of developmental psychology and cognitive neuroscience are much more emphasized which are also relevant for mathematics and science. Some of the cognitive abilities are described in detail in Chapter 1 of the volume on science but these abilities are also to be developed in mathematics. The second chapters of the volumes are devoted to the issues of how knowledge is applied and the general conclusions of any chapter can be used in the other two assessment domains. Chapter 3 also deals with practical, curricular issues in every field related to historical traditions of the Hungarian education system and the current practice. At the same time, in the selection and the arrangement of the content of education there seems to be a demand for following advanced international trends and making use of results achieved in other countries.

In line with these principles, the nine theoretical chapters altogether are considered to be the theoretical foundation of the diagnostic assessment system. In all three domains, we can draw on the background knowledge provided in the theoretical chapters without going into details of the identical issues in the parallel chapters.

The main aspects of the development of the frameworks are outlined in the first part of this chapter. First, the means used to describe the objectives of education and the content of assessments is described then our solution to the detailed description of the content of diagnostic assessments is presented. Further on it is shown in what ways these principles are applied in the elaboration of the frameworks of mathematics.

Taxonomies, Standards and Frameworks

We have made use of various resources during the creation of the frameworks for diagnostic assessments. We have attempted to give a clear-cut definition of the educational objectives and the contents of the assessments. First an overview of the systems describing contents is given then the methodology we have applied is shown related to these systems.

Taxonomies

The attempt to define educational objectives dates back to the 1950s. It was this time that the taxonomies by Bloom et al. were published, which later on have predominantly exerted an influence on pedagogy. What actually triggered the interest in taxonomies was the widespread discontent with the lack of clarity and precision in the description of curricular objectives on the one hand and the cybernetic approach making its way into education on the other. The demand for regulation has emerged, which required feedback and the prerequisite of feedback is to measure the distance between the aims that were set and the results achieved. By comparing the aims and the actual situation the deficiencies can be revealed and intervention can be planned accordingly. During the same period of time pedagogical evaluation became more pronounced and the widespread use of tests also required a clear definition of the object to be measured.

Taxonomy as a matter of fact is a structural framework, which serves as a kind of guide-line to arrange, to systematize and to classify things, in this particular case knowledge to be attained. It is like a chest of drawers with labels indicating what to put into them or like a table in which the headers are filled in showing what in the various columns and lines can be found. In comparison with the earlier general descriptions, making plans based on standardized systems represented a great progress and those involved with outlining concrete objectives to be attained in various subjects were in a way forced to consider their expectations resulting from instruction.

What exerted the greatest impact was the taxonomic system of the cognitive system published first by Bloom et al., 1956, which opened up new perspectives for designing curriculum and evaluation systems. The behavior patterns expected from students is described in terms of concrete and observable categories by the taxonomic system. What emerged as the greatest novelty was the six levels of frameworks built on each other, which could be uniformly used in every area of knowledge. Besides going into details, getting down to facts and accuracy represented enormous steps forward compared to earlier methods. Making use of the same detailed description both for planning learning processes and devising assessment devices implied a further advantage. This way it is taxonomy of objectives and evaluation.

It was in the US that the Bloom taxonomies exerted the first direct influence and later on they served as a basis of the first international IEA surveys. The hierarchy of knowledge assumed in the taxonomy was not confirmed by the empirical studies in every detail. Moreover, the underlying theory of the Bloom's taxonomy, the behaviorist approach was pushed into the background in the psychological foundation of education giving way to other paradigms, primarily cognitive psychology. Thus the initial cognitive taxonomies were rarely applied. The similar taxonomies in affective and psycho-motor areas were elaborated later on and even if they had been applied in several areas, their impact was not that predominant as that of the cognitive ones.

Taxonomies as principles of systematization are "empty systems" which are not concerned with concrete content. In taxonomy handbooks content is presented only by way of illustration. Thus, for example, when the six levels in *Bloom* taxonomy, *knowledge, comprehension, application, analysis, synthesis and evaluation* are used to describe objectives to be attained in a particular field of geometry, then it has to be clearly given what level of knowledge, comprehension and application is expected in geometry, etc.

Following the original taxonomies or revising them, currently more up-to-date handbooks have been produced in order to assist the description of latest systems of objectives (Anderson & Krathwohl, 2001; Marzano & Kendall, 2007). These works carry on the tradition established by Bloom, the operationalization of the objectives, and breaking down of knowledge into concrete measurable units. Methodologies established during the elaboration of taxonomies can serve as useful methodological resources for setting up standards.

Standards in Education

An impetus was given to setting up standards in the 1990s, especially in English-speaking countries where the normative documents regulating the content of education did not actually exist. In some countries, to say the least, in every school it was taught what was decided locally. Under these circumstances the options of educational policy narrowed down and the chances for improving the output of the school systems seemed rather poor. This is why the centralizing efforts were gaining ground and the aims and objectives of education were centrally set up either at local or at national level.

Educational standards are detailed descriptions of what students are expected to master, which in contrast to taxonomies as systems are concerned with concrete content. The standards of different domains are generally prepared by different professional teams and therefore according to the particular branch of knowledge, diverse solutions of forms can be applied.

Standards are normally devised by teams of specialists, who draw on the most up-to-date theories and scientific achievements. In the US for instance the professional association of mathematics teachers, the National Council of Teachers of Mathematics, (2000) created the standards for the 12 grades in public education. These standards generally describe what level of knowledge is to be achieved by students in various subjects after completing a grade.

The elaboration of standards went hand in hand with their application, similarly to the taxonomies both in assessment and the educational process. Several handbooks have been published which provide detailed methodologies of setting up and applying standards. However other aspects are emphasized than those in the taxonomies. Standards exert an influence directly in education (see Ainsworth, 2003; Marzano & Haystead, 2008) and evaluation seems to be only secondary to them (for example, O'Neill & Stansbury, 2000; Ainsworth & Viegut, 2006). What standard-based education actually means is that there are detailed, standardized requirements whose attainment can be expected from students of certain age groups.

Standards and standard-based education are not a complete novelty to Hungarian and other experts in education who have gained ample experience in centralized education systems. In Hungary, prior to the 1990s the contents of education were determined by a central curriculum and all the text-books were based on it. Every student of the primary schools studied the same teaching material and in theory every student had to meet the same requirements. In some fields, such as mathematics and science the standardized curricula had been the outcome of experiences of several decades, whereas other fields were exposed to political and ideological pressure. The trends of the 1990s were highly influenced by the earlier Anglo-Saxon model and the pendulum swung back, so the National Core Curriculum contained only minimal central requirements. This process was actually in contrast with what happened in other countries in the same period of time. By way of comparison it is worth mentioning that the printed volume presenting American mathematics standards (National Council of Teachers of Mathematics, 2000) is about the same in size than the first version of the Hungarian National Core Curriculum defining all domains of education published in 1995. In the meantime, the Hungarian national Core Curriculum has become even shorter.

Standards and standard-based education imply not merely standardization or centralization, but the professional arrangement of the contents of education based on research results. In this respect it is different from the earlier Hungarian central regulation to which this description applies only in some respects. The new kind of standards has become accepted even in countries where there were central curricula even earlier. For example, in Germany where the educational contents were determined at the level of lands earlier research into the development unified standards has begun (Klieme et al., 2003). The solid theoretical background is considered to be the essential feature of standards. Thus the elaboration of standards and standard-based education has triggered world-wide research and development.

During the preparation of the frameworks of diagnostic assessments we have relied on the theoretical implications of standard-based education on the one hand and contents and forms available in particular standards. In line with the traditions of setting up standards the characteristics of various content and assessment fields have been taken into consideration and we did not attempt to come up with totally identical solutions in forms in the description of the contents of reading, mathematics and science.

Our frameworks are different from standards because they do not set requirements and expectations. What the frameworks and standards have in common are the detailed and concrete description and the demand for solid theoretical background.

Frameworks

In line with the English usage the term framework is used for the detailed descriptions outlined in the present project. The frameworks of the assessments are similar to standards as they also provide detailed and systematic descriptions of knowledge. In contrast with traditional curricula it is not determined in the frameworks what and at what level should be taught and learned. Even the requirements to be achieved are not defined, although the content descriptions implicitly imply what can or should be achieved at the given domain.

The best-known frameworks have been prepared for international surveys. In case of assessments carried out in several countries laying down the requirements was of course out of the question. In this case what is presented by frameworks is what can be assessed and what is actually worth assessing. In outlining the content various aspects can be taken into consideration. In early IEA assessments the curricula of the participant countries were the starting point, namely what actually was taught in the given field.

In the main domains of the PISA assessment frameworks, it is presented what applicable knowledge is needed by the fifteen year-old young people of modern societies. In this respect the application of knowledge, and the needs of modern societies as well as the typical contexts of application play an essential role in the elaboration of frameworks related to the application of knowledge in various disciplines and school subjects.

Thee third approach mainly relies on research results related to learning and knowledge based on developmental and cognitive psychology. This aspect has been prevalent in the cross-curricular areas which were concerned with not one (or some) school subject. This kind of assessment was for instance in the fourth area of PISA 2000 where learning strategies and self-regulating learning were placed in the foreground, whose frameworks were essentially based on psychological research results related to learning (Artelt, Baumert, Julies-McElvany & Peschar, 2003). Students' attitudes can also be described according to psychological principles. The study of attitudes was carried out in most of the international assessments, and it was highly important in the PISA 2006 science survey (OECD, 2006). Similarly, psychological research has revealed the structure of problem-solving, which was the innovative assessment area of the PISA 2003 (OECD, 2004). The latest cognitive research results are to be used in the PISA 2012 dynamic problem-solving assessment.

The frameworks outlined for diagnostic assessments (see Chapter 5) have drawn on the frameworks of international assessments. They resemble PISA frameworks (for example, OECD, 2006, 2009) in a way that by focusing on three main assessment areas they lay down the foundations of the assessments of reading, mathematics and science. Whereas PISA frameworks have focused on one age-group, namely the fifteen-year-old young people, our frameworks focus on six grades, younger students, and developmental aspects are much more emphasized.

PISA frameworks have been prepared for a given assessment cycle, and although the frameworks are updated in every period the consecutive assessment cycles overlap. The PISA frameworks comprise the overall assessment process, from defining the domain, and organizing the domain to reporting scales showing the results. Relying on the above assessment process our frameworks imply the definition of the domain, the organization of the domain and the detailed description of the contents. The main dimensions of the assessments and the assessment scales are shown but in the current phase we are not concerned with the levels to be achieved on the scales and the quantitative issues of scales. With regard to developmental aspects drawing up scales needs further preliminary studies and empirical data.

Multidimensional Organization of the Assessment Contents

Over the past decade, innovations in education have been mainly integrative. Competencies themselves can be seen as the complex units of various elements of knowledge supplemented with further affective elements according to some interpretations (see e.g. Hartig, Klieme, & Rauch, 2008).

Competency-based education, project method, problem based learning, inquiry based learning, content-based development of skills, content-based language teaching and many other learning and teaching methods fulfill several objectives at the same time. Knowledge gained in this integrative way can be assumed to be more easily transferable and applied in a wider range. Summative achievement tests as well as PISA tests and competency assessments in Hungary are based on similar principles.

Other assessments are needed when learning problems are to be prevented and deficiencies hindering students in future achievements should be identified. When assessment results are used to identify the necessary intervention, it is not sufficient to prepare tests providing general indicators of students' knowledge. It is also not sufficient to find out whether the learners are able to do a complex task. The reasons for eventual failure should also be revealed as to learners lack in basic knowledge or the operations of thinking are not completely available, which are required for turning pieces of knowledge into logical chain of conclusions.

Diagnostic assessments need a detailed description of students' knowledge, which is why an analytic approach was used in contrast with the integrative approach in instruction. At the same time, the assessments assisting learning should also be adapted to the concrete educational processes. In line with these requirements the techniques of diagnostic and formative assessments are being worked out, which also draw on the results of summative assessments based on large samples. Moreover several fresh elements are introduced into the assessment techniques (Black, Harrison, Lee, Marshal, & William, 2003, 2005; Leighton & Gierl, 2007).

Several lessons can be drawn from the earlier achievements in similar fields for outlining the frameworks of diagnostic assessments, especially assessments conducted in early childhood (Snow & Van Hemel, 2008) as well as formative techniques for the initial stages of education (Clarke, 2001, 2005).

What we have considered most relevant is the multi-faceted, analytic approach emphasizing the psychological and developmental principles. The earlier paper-based formative and diagnostic tests have had their limitations. We use on-line tests which make it possible to conduct more detailed assessments more frequently, and so the frameworks should also be adjusted to these possibilities.

The Aspects of Organization of the Content to be Assessed

The content of the assessments can be organized according to three aspects. This way the content of the assessment makes up a three-dimensional cube, which is shown in Figure 4.1. The detailed description of the content however needs the linear arrangement of the blocks of this three-dimensional cube. The elements of the block can be listed in various ways depending on the way it is cut, according to which dimensions are cross-section produced first, the second and the third time. Here the aspect is shown, which seems to be more suitable for the requirements of diagnostic assessment.

The aspect highlighted first is itself a multi-dimensional system, which represents the three main dimensions of our analysis, namely the psychological (thinking), social (application) and disciplinary (subjects) dimensions. The developmental scales have been set up for these dimensions in the three main areas of assessment, such as reading, mathematics and science. (A more detailed description can be found in the next section.)

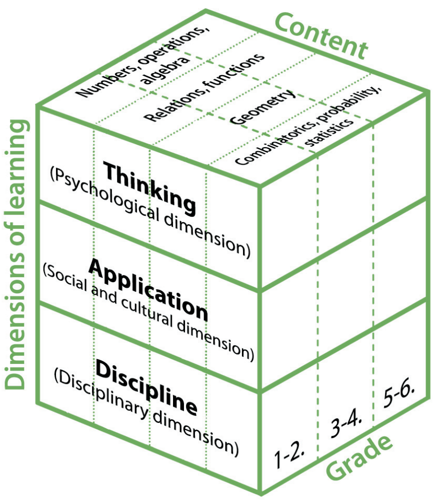

Figure 4.1 The multidimensional organization of the content of assessments

The second aspect is development. Accordingly the six grades were divided into three two-year stages such as grade 1-2, 3-4 and 5-6. Since the six grades are seen as a uniform developmental process, this division serves only as a practical solution to organize the content. Without empirical evidence, attachment to age group (grades) can only be hypothetical and approximate.

Finally a third aspect is the range of contents available in the given assessment area. The blocks of contents broken down in this way provide the units of detailed frameworks. Due to combining various aspects the increase in the values of certain aspects may result in a combinatorial explosion. Thus we have to be careful with the number of concrete assessment contents. Differentiation according to three dimensions of learning, three age-group blocks and four main contents in case of mathematics gives 36 blocks. This number would rapidly increase by differentiating further areas.

Scales of Diagnostic Assessments, Psychological, Application and Disciplinary Dimensions

Based on empirical results of earlier studies, a model was created, whose three dimensions are in accordance with the three main objectives of education. These objectives can be traced back to the history of schooling and are actually in line with the main trends of assessment of school achievements (Csapó, 2004, 2006, 2010).

Cultivating the mind and the development of thinking are kind of objectives which imply not external contents but rather internal characteristics. To put in modern terms, this is the psychological dimension. As it has been mentioned above, in the PISA surveys this dimension is also included. In several assessment areas the content of assessment was interpreted in psychological terms. In this respect in mathematics it is examined whether mathematics is developing thinking, general cognitive abilities to a degree it can be expected.

Another age-old objective is that usable knowledge should be provided by education which can be applied not only in schools. This aspect is the social dimension, and it covers the utilization and application of knowledge. The term has a lot in common with the transfer of knowledge which implies that knowledge gained in one context can also be applied in another one. The degrees of transfer can be characterized by the transfer distance. Any solution of task is the application of mathematical knowledge when what has been learned in one field of mathematics is applied in another field (near transfer) or beyond mathematics in other subjects or in practical context (distant transfer).

The third frequently mentioned objective is that students should obtain the essential components of the knowledge accumulated in science and art. This objective is being attained when the learning process is determined by the aspects and values of a particular branch of science, and this is the dimension of school subjects and disciplines. Over the past few years there have been attempts in education to counterbalance the earlier one-sided approach based on the branches of science. The competency-based education and large scale assessments focusing on application have pushed the considerations of the branches of science in the background. However in order that the teaching material should be a coherent, standardized and in this way available system in accordance with branch of science, the components of knowledge are also to be acquired, which contribute to application and the development of thinking indirectly. The establishment of the scientific validity of concepts and exact definitions are parts of this kind of knowledge.

The three-dimensional model implies that the same content, perhaps with a shift of focus, can be used for writing tasks in each dimension. By way of illustration we refer to the combinatorial reasoning, whose basic level is being formed in childhood through the interaction with the environment. This kind of thinking is improved by school activities and thus the levels of the ability of combination, a part of cognitive development can be measured. Or tasks can also be prepared in which combinative thinking and knowledge gained in combinatorics are to be applied in everyday situations. Finally it can be checked whether students are aware of variation and combination and they also know how to obtain formulas required for their calculations. This latter is a kind of knowledge that cannot be attained by activities encouraging cognitive development but only through proper knowledge of mathematics.

Among the three assessment areas mathematics is rather special with respect to the fact that the evolvement of mathematic thinking, especially in the early stages of education is closely related to the general cognitive development. In every field of mathematics, carrying out operations and thinking have an essential part, which is why the three dimensions are not separated from each other so much as it happens in the other areas of assessments. Thus it frequently occurs that the aspects of various dimensions emerge even in the same task.

Abilities in Mathematical Reasoning

In Chapter 1 of the present volume the components of mathematical reasoning are outlined on the theoretical base provided by Piaget and Vygotsky. Moreover a system of skills is proposed, which is comprehensive and at the same time specifically mathematical. Being comprehensive means that the mathematic thinking processes frequently designated in various ways are described according to four fundamental abilities in mathematical thinking. The system of abilities outlined in the chapter is still specifically related to mathematics. Independently of the structure of the science of mathematics and also social expectations, integers, rational numbers, the complete system of additive and multiplicative thinking patterns provide the description of mathematic thinking. In the following a further link will be set up between the theoretical chapters and the detailed frameworks through two sets of concepts, which have had a key role in international research over the past two decades.

Problem-Solving

In a major part of research on mathematical reasoning, it is considered to be a field of general problem-solving. Normally a task is called a problem if no algorithm is available to follow in the solution process, but the task requires conscious planning, monitoring and checking. When answering questions such as “how can you measure the length of the school-yard by using your steps?” or “how many liters of water goes in your bathtub?” students break down the problem into parts and analyze it and sum up the steps of problem-solving and solve the problem. The awareness of mathematical concepts and symbols and the development of mathematical skills and abilities are the precondition and assistance for the development of problem-solving in this sense.

Apart from mathematical abilities described in Chapter 1, other reasoning processes are also involved in mathematical problem-solving. This is what makes it possible to apply mathematical operations in new, unfamiliar situations in solving problems different from the routine tasks which need a lot of practice.

In the study of mathematical problem-solving, mainly word problems are used and the process of their solution is analyzed (Csíkos, Kelemen, &

Verschaffel, 2011). Word problems as the natural means of the development of problem-solving are actually used as early as in grades 1-2. However the precondition of solving word problems is proper comprehension, thus the length and the linguistic complexity of the text should be adjusted to developmental level of the learners. Initially word problems are described by two or three simple sentences. Another stage is when task is done not after listening but reading. As it can be seen in grades 3-4 word problems, originally meant to assist problem-solving by the various means of mathematical modeling of reality, more and more have become the means of improving counting ability. This is why understanding the text of the task and exposing the problems in an accessible way are essential.

An important method for improving problem-solving is putting the thinking processes into words, and asking "why" questions (Pólya, 1945). Discussing internal, mental images and drawings learners produced about the task (with great variety, according to their mental models) may contribute to the development of strategic and meta-cognitive components of problem-solving.

Mathematical Skills and Abilities

Over the past two decades, major efforts have been made to define skills and abilities related to mathematics either in terms of development or assessment. One of the trends in the research into intelligence has attempted to reveal the structure of intelligence based on the differences in the measurable psychological characteristics. It was Carroll (1993) who summarized the findings of this line of research and then he made an attempt to describe mathematical skills in the system of abilities of intelligence. According to Carroll (1998), several components of fluid intelligence can be directly detected in mathematical achievements, for instance general sequential reasoning, quantitative reasoning and the so-called Piagetian reasoning. Carroll also has highlighted several components of crystallized intelligence as the key elements of mathematical abilities.

Linguistic development is also highly relevant as people tend to count in a particular language and the designation of numbers in various languages influence counting skills as well. Oral and written comprehension skills are obviously needed to understand mathematical word problems. Some of the components of counting ability can be found in the factors of intelligence, such as the general processing speed and number facility, which the measurable components of thinking are as a matter of fact.

In Hungary, there are research programs, which aimed at studying the establishment and early development of counting skills: PREFER and DIFER (Nagy, 1980; Nagy, Józsa, Vidákovich & Fazekasné, 2004).

Reasoning abilities are highly relevant in mathematical thinking. Hungarian diagnostic assessment programs have been involved with the following five thinking abilities: inductive (Csapó, 2002), deductive (Vidákovich, 2002), systematizing (Nagy, 1990), combinative (Csapó, 1998), and correlative (Bán, 2002). These abilities, which are also important in mathematics, can be assessed and developed in varied mathematical contents as well.

The Domains of Applying Mathematical Knowledge

During the development of mathematical concepts there is a continual interaction between the observed phenomena and the emerging mathematical concepts. As Rényi (2005, p. 39) puts it:

"When children are taught to count first they count pebbles or sticks. Only if children are able to count pebbles or sticks will they be able to realize that not only two pebbles and three pebbles are five, but two something and three something are always five something, i.e. two and three are five."

Understanding and using mathematical concepts can happen at several levels. In cognitive approach, understanding is quite often seen as eliminating cognitive dissonance (Dobi, 2002). The level of understanding, when mathematical concepts themselves are considered (as Rényi did), represents a higher level of understanding. This higher level of understanding is labeled 'reflective thinking' by Skemp (1975).

When the application of mathematical knowledge is assessed, children solve word problems in which concepts and mathematical phenomena known from everyday experience also appear. Since ancient times at least three functions of the word problems have been intertwined.

(1) Putting mathematical knowledge into words, namely improving and practicing mathematical skills and abilities through word problems. In this case the wording of the tasks is familiar and student-friendly but not necessarily is concerned with practical problems. These kinds

of tasks are called "educational examples" by Julianna Szendrei, 2005). Routine examples or as we may call them practicing word problems primarily aim at developing and practicing mathematical skills and abilities and the concrete content of the wording can be almost freely replaced.

(2) Mathematical word problems can serve as a means of mathematical modeling of the world. The clerks in ancient Egypt and the merchants in Venice during the Renaissance were trained through mathematical tasks containing everyday situations and problems they had to cope with in real life.

(3) Recreational and riddle-like mathematical word problems also date back to many thousand years. The archetype of this kind of word problems is the nonsense sort of riddle asking "how old is the captain" and also the riddles in which the person trying to find the solution is expected to figure out what actually was meant by the task. The obviously incomplete tasks and also what is called insight problems in the psychology of creativity also belong to this field (Kontra, 1999).

The above-mentioned functions of word problems are quite often interrelated. It can happen that for some students in a given age-group a task can be a word problem to be routinely solved, while for others it can be a means of mathematical modeling of the world. Moreover depending on the way the task is set, students can consider the same word problem a practicing word problem or a problem whose solution mobilizes their resources and makes it possible to choose various mathematical models. In the chapter dealing with the application of mathematical knowledge several cases are shown in which learners who failed to realize that a word problem is actually a routine example got into disadvantageous situation. Thus it can be said that information on the process of the solution of word problems is part of mathematical knowledge. The role of teachers is emphasized by the fact that students in various grades acquire different socio-mathematical norms as to the nature, process and ritual of solving tasks.

Realistic Word Problems

As it has been described in the theoretical chapter on the application of mathematical knowledge, there is a group of word problems whose primary function is not the wording of some mathematical operation or knowledge components but rather assisting the mathematical modeling of familiar components of knowledge which are accessible in their imagination and experience. Where can the the border between routine examples and realistic word problems lie?

Actually no tasks can be called realistic or not realistic. To make a distinction between them several factors need to be taken into account. It takes at least three factors to consider a word problem realistic.

(1) The role of things and properties in the word problems: if the characters, phenomena and properties are essential part of the word problems inasmuch as their changes lead to essential changes in the process of solution, then probably it can be considered a realistic word problem.

(2) The relationship of the things in the word problem and students' knowledge. The adjective 'realistic' in the original sense refers to the fact that the things in the word problem are imaginable. It is not a requirement that only everyday objects should be found in realistic word problems. Even a combinatorial word problem involving the seven-headed dragon can be realistic.

(3) In a classroom setting it is the socio-mathematical standards that determine to what degree the ritual and regulation is binding in the steps of the solution of word problems. From this point of view what may indicate that a word problem is realistic is that the 'familiar' algorithm fails to work in the solution.

It frequently happens in case of realistic word problems that even gathering data and selecting the operations to be carried out requires finding a mathematical model implying planning, monitoring and control processes.

Authentic Word Problems

Within the set of realistic word problems a specific group called authentic word problems can be found. In the theoretical chapter the features of authentic word problems were defined and in the detailed assessment frameworks they are described and illustrated by examples.

Authentic word problems, which are often intransparent problems, rely on l students' experience and activity. It is a specific feature of authentic word problems that by emphasizing students' activities they are much more motivated and involved. One superficial characteristic is that the texts in authentic word problems are longer, in which a real problem situation is described from mathematical point of view by means of either redundant or missing data. It may also be a characteristic of authentic word problems that students are asked to set a task related to the problem described. In the process of problem solving what is essential is the fact that in authentic word problems there is no direct, obvious algorithm to the solution, thus real mathematical modeling is needed, during which different activities take place. Activities which can also be observed are gathering data from external resources or by means of discussion, measurements, debates and conversations drawing on students' knowledge gained previously.

It can happen several times, as it happens in authentic every problems as well, that there is no single, well-defined solution to the problem, however from pedagogical point of view the process of dealing with mathematics, including planning, monitoring and control can be quite often considered as the solution of authentic problems. The solution of authentic word problems sometimes takes place in noisy team work, which can be very different from the traditional mathematical classroom setting approved by both lay people and teachers.

It was George Pólya (1945, 1962) who came up with one of the first models of mathematical problem-solving. The steps of successful mathematical problem-solving described by him can mainly be found in the solution of realistic and also authentic problems. The questions raised by Pólya, which these days might as well be called meta-cognitive questions, apart from the mathematical characteristics of the problems are concerned with the relationship of the person solving the problem and the mathematical problem. "Could you describe the problem in your own words?" "Can you come up with a figure or diagram that could be conducive to the solution of the problem?"

Domains according to the Science of Mathematics

In the diagnostic assessment of mathematical knowledge tasks are normally related to various domains of mathematics. As it was presented in chapter 3 the domains of mathematics in education are in accordance with the current structure of mathematical science. Different domains are put in the foreground in different grades and the historical development of some of the domains is diverse in Hungarian public education.

Numbers, Operations and Algebra

Numbers, operations and algebra are considered the basic foundations of teaching mathematics. In grades 1 and 2 the most time and energy is devoted to the development of counting skills. This domain includes the development of the concept of numbers, the extension of numbers, the acquisition of the four basic mathematical operations, and also the preparation for algebraic thinking by using signs instead of numbers. Moreover the requirements to apply mathematical knowledge also include modeling multitude observable in reality and everyday phenomena described in terms of the basic mathematical operations.

It is also essential to take into consideration the pedagogical implications of Dehaene's triple-code theory (1994). The names of numbers (primarily natural numbers), the written form of Arabic numbers and the interrelationships of mental quantity representation attached to a given number make it possible for students to obtain an established number concept. Even before they go kindergarten children know the name of some numbers and in case of smaller quantities they use them in a meaningful way, for instance two ears, three pencils. The written form of numbers is attached to numerals mostly at school.

Research results concerning the development of quantity representations attached to numbers show that in grade 2 the mental number line in case of natural number less than 100 is rather exact and linear (Opfer & Siegler, 2003). This makes it possible that by the end of grade 2 dealing with the set of numbers less than hundred that the written form of numbers, the oral naming of numbers and some sort of quantity representation are related to each other.

Basic mathematical operations are described in terms of the general principles of developing and improving skills. The stages of development are well-known, the familiar breaking-down points which hamper students from getting to 7 after 6 or , to 17 after 16 (Nagy, 1980). It is also shown by data that sometimes the basic counting skill can become too automatic in the lower primary grades. This fact can be traced back in the quantitative comparison of word problems and reality, or in the lack of it. When algebraic signs are introduced to this age group simple geometric shapes, such as squares, circle, semi-circle are used to designate unknown quantities.

Relations and Functions

Recognizing rules and patterns in the surrounding world is considered as one of the characteristics of reasoning. In the field of mathematical thinking the recognition and description of relationships can belong to various areas depending on the data and phenomena and whether the relationship is seen as deterministic or of probabilistic nature.

In the mathematical definitions of relations and functions, sets and mappings can be found. Both sets and mappings are basic mathematical concepts, that is why it is highly important to attach them to students' everyday experience, ideas and basic concepts. In dealing with this topic special difficulties may arise as to what degree the abstract mathematical concepts of relations and functions can be associated with visual images such as the tables of "machine-games" or the curves in the Cartesian two-dimensional coordinate system.

In the National Core Curriculum most of the requirements related to functions have not been linked to particular age groups. In terms of our assessment framework it implies that the development students' thinking is to be assessed through a well-defined system of tasks. For instance the requirement in National Core Curriculum namely "recording the pairs of data, triplets of data of quantities changing simultaneously: creating and interpreting functions and sequences based on experience" applies to every age-group of public education. Regarding assessment frameworks however it should be decided how to operationalize the components of knowledge based on each other and into which age-group sections they should be put. With respect to this requirement the following questions seem to be relevant. What kind of quantities simultaneously changing should be included into the tasks? In which grades the pairs of data and triplets of data should be presented? In what ways students are expected to provide the relationship between the data? What kind of vocabulary is expected in various grades to describe character and closeness of the relationships between the variables? Besides having raised these specific questions we still consider the topic of Relations and functions highly important in the development of proportional reasoning and more generally of multiplicative mathematical reasoning.

Geometry

Geometry, like the topic of *Numbers, operations and algebra,* has been traditionally embedded in the curriculum. As it is shown by the IEA international comparative curriculum analysis, in Hungary a large section is devoted to geometry in the mathematics curriculum (Robitaille, & Garden, 1989).

In the National Core Curriculum, it is for example, orientation which is emphasized among the objectives, values and competencies, and it can be defined as a sub-section of geometry. Geometry and the topic of measurements are suitable for attaining the objectives to orientate in space and in quantitative relationships of the world.

Every aspect of cognition is touched upon when geometry is covered, and the various approaches of creativity and activities that are autonomous, based on students' own plans, in line with given conditions, especially in the initial stage of geometry teaching are to be highlighted. Furthermore, creative activities contribute to the development of cooperation and communication.

Space and plane geometric perception is being formed by children's concrete activities and also through materials gained by various techniques representing the world, as well as models, for example, objects, mosaics, photos, books, videos, computers.

In the NCTM standards mentioned above in all grades the field of measurement is separated from geometry. In our view the principles and requirements related to measurement should be included into geometry. The two lists below, in which the NCTM major requirements are pointed out, make it clear that in the Hungarian mathematics education the two areas are getting along well with each other under the umbrella of 'geometry'.

The aims and objectives in geometry outlined by NCTM standards for this age-group:

(1) The characteristics and the recognition of two-and three-dimensional geometric shapes, their designation, building, drawing, description, discussion skill in geometry.

(2) Orientation in space and time, description, designation and interpretation of relative positions in space, application of knowledge.

(3) Recognition and application of transformations such as shifting, revolving, reflection as well as recognition and creation of symmetric shapes.

(4) By means of spatial memory and visual memory producing mental images of geometric shapes, recognition and interpretation of shapes represented in various perspectives, and making use of geometric models in problem-solving.

In the topic of measurement the objectives and requirements of NCTM Standard are rather similar to that of the framework curricula.

(1) Understanding the measurable properties of objects, units, systems and processes, such as length, weight, mass, volume, area and time, comparison and organization of objects according to these properties, measurement by means of accidental and standard units, selection of the tool and unit suitable for the properties.

(2) Using techniques, tools and rules suitable for the definition of sizes (comparison of measurement, selection of units, and the use of measurement tools).

Combinatorics, Probability and Statistics

In the first six grades of public education, the objective of teaching combinatorics, probability and statistics is mainly to gain experience. Accordingly the curriculum requirements focus on the development of basic skills rather than expanding knowledge. The concrete, future-built knowledge in this domain needs the foundation of combinative and probabilistic reasoning based on experience.

In the initial stage combinative thinking is primarily established by acquiring the importance of systematization. At first what children are interested in is not the number of options, but rather seeking and coming up with options. We attempt to be demanding in two ways. Focusing on particular conditions throughout the task on the one hand and checking what they have done, whether they have produced anything like this, whether the new thing is different from the rest. In secondary school curricula and GCSE requirements probability is much more in the foreground than before. This is why the topic needs more thorough preparation in lower primary grades. However there is a great difference between *development of probabilistic approach and the calculation of probability.* Theoretical calculations are highly different from experience gained from experiments. The growing awareness of putting down data is indispensable for the deeper understanding of the topic of statistics. At the beginning what happens more frequently is considered to be more probable and only at a later stage is it modified and seen that what can happen in several ways is more probable even if it is not supported by experimental data. Accordingly the curriculum development and assessment are also based on gaining experience. The concepts of "certain", "not certain", "probable" and "possible" can be best established through games, activities and gathering everyday examples.

Summary and Future Objectives

The detailed frameworks of mathematics are as a matter of fact the starting point for the elaboration of diagnostic assessment system. This is the initial stage of a long developmental process, in which the conception of assessment has been outlined, the research results have been summarized and the contents needed for the tools of assessment have been described in detail.

The theoretical background and the detailed frameworks can eventually be upgraded relying on various resources. At this point, due to time limitations it was not possible to organize discussions with out-of-school agents (decision makers, local stakeholders). These Hungarian and English language volumes will be available for the wide scientific and professional public. In the first stage of upgrading the feedback from professional circles will be processed and used.

The second, so to speak continuous stage is relying on the latest scientific results. In some fields rapid progress is taking place, for instance the research into early childhood learning and development. Understanding and operationalization of knowledge, skills and competencies can be found in several research projects. Similarly intensive work has been going on in the field of formative and diagnostic assessment. These research results can be used for reshaping the theoretical background and improving the detailed descriptions.

The development of the frameworks can actually make use of their application. Data, produced continually by the diagnostic system can be used for the study of theoretical framework as well. The system outlined at this point is based on our current knowledge and the organization of the content and linking age-groups roughly to it can be said to be a hypothesis. It will be shown by assessment data what students know at what age and further experiments are needed to find out what level students can achieve by means of efficient organization of learning.

Analyzing the connections between various types of tasks can reveal the connections between different measuring scales used for monitoring students' development. In the short run, the tasks can be analyzed which of them belong to specific scales and which are the ones that can belong to several dimensions. Diagnostic assessment data are also important in connecting the results longitudinally. In the long run the diagnostic value of the particular problems can also be investigated which knowledge domains can determine students' achievement in the future.

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5

Detailed Framework for Diagnostic Assessment of Mathematics

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The structure of the detailed assessment frameworks of mathematics is based on the theoretical background explained in the introductory chapters. In this chapter a three-level structure is presented according to the following scheme. The primary structure of the chapter is determined by three dimensions of learning mathematics. Within this chapter, first the psychological principles are highlighted showing that only such mathematics teachning can be successful which adjusts to the natural processes of cognitive development and improves reasoning. The second part of this chapter describes mathematical knowledge according to its application, and the third part is built according to a pure mathematical disciplinary approach. In the case of mathematics, the three dimensions of knowledge are mutually intertwined, and – as emphasized in previous chapters – distinguishing them serves the purpose of detailed diagnostic assessment. Certainly, the three dimensions appear in teaching in an integrated way, almost unnoticed and the problems of different dimensions are manifested parallel during the assessment.

The second aspect of the structural division is the school years. Due to the big differences between the pupils the age intervals can only be approximate, while by assigning the frameworks to several age groups the principle of interdependence and development is emphasized. The third basis of structuring is determined by the different fields of mathematical science. Since the developmental processes arch several grades, these contents appear at different levels in every grade.

It follows from the above-described structure that this chapter is divided into 36 parts. To every age group 12-12 units are belonging; the different fields of mathematics are represented by 9-9 sub-chapters and also 12-12 sub-units belong to the three knowledge dimensions. The theoretical chapters describing the different knowledge dimensions (the first three chapters of the present volume) contain the criteria of selecting the age categories and knowledge areas. It comes from the nature of the development processes that the focal point of development in certain areas is earlier, in other comes areas later. Therefore the following 36 parts are not equally proportionate or detailed. The further clarification of the details is however only possible after conducting surveys and possessing empirical data.

Diagnostic Assessment of Mathematical Skills

Detailed Assessment Frameworks of Grades 1-2

Numbers, Operations, Algebra

During the development process, in the lower grades we get from the well planned concrete activities, from the reality experienced by the learners to the more abstract formulations by drawings, words, signs and symbols through visual and audio-visual representations of real life. The correct harmonization of reality, concept and symbol (sign), their bringing in compliance with each other is the result of a lot of activities. The development of the system of skills indicating the competent use of whole numbers begins already in the preschool age. The fact that it is clear for the learner beginning the school that the bigger quantity is represented by bigger number is an indicator (among others) that the learner is at good development level concerning whole numbers as elements of mathematical reasoning.

A typical preschool exercise:

Draw more circles on the right side than you can see in the left side frame.

In the first grade we go further in the questions, instructions:

1. *Draw three circles more on the right side than you can see on the left side.*
2. *Describe, by the language of arithmetics, what you see on the figure. (*Solution*: 3+3+3=9; 3+6=9; etc.)*

In the second grade the mathematical content of the questions is extended further:

1. *Add more circles on the right side so that you get a total of 18 circles on the drawing.*
2. *Write additions, subtractions in connection with the figure. (*Solution: *18 – 3 = 15; 3 + 3 + 12 = 18; 15 – 3 = 12; etc.)*

3. *Surround the circles with red colour so that the same number of circles chould be within every enclosure. (*Solution*: 1×18 circles or 2×9 circles or 3×6 circles or 6×3 circles or 9×2 circles or 18×1 circles)*

The common experiences and collective mathematical activities create a kind of shared reference basis for the class/group. The richer and more mobile this reference basis is the more sure it is that the same image, sequences of actions, memories, ideas will be evoked in every pupil by the questions, statements and other formulations.

Numbers

Children coming from the kindergarten have memories about comparing objects and pictures, about studying characteristics, looking for relations and about their efforts to formulate relations, in accordance with their developmental level. The well prepared and diverse activities continue in the school, the content elements of concepts are made understandable. In this way the pupils understand and use appropriately the relations of more-less (for example, by one to one mapping), same quantity (for example, by putting into pairs, which pairing is the method of mastering this relation), smaller-bigger, longer-shorter or higher-lower (for example, by comparative measurements), etc. The relation symbols (>; <; = symbols) are given names related to the children's environment, fairy-tale world (for example, the mouth of the fox opens to that direction because he sees more chicken there), but in certain cases the "relation symbol" name is also used. (The early introduction of mathematical expressions has to be treated carefully, because due to this they may be imprinted incorrectly (for example, with narrower content) and this may be disadvantageous, leading to a lack of understanding later).

The sequences of observations, comparisons enable the pupils to make identifications, to recognize and name the important characteristics contributing to differentiation, to make abstractions gradually (for example, making them recognizing the differences between two drawings of small dogs draws the attention not only to the physical contours (for example, pulling off or lifting up the dog's ear), but also to the differences expressing emotional/mood status (for example, the dog is sitting quietly or muscles taut and face angry, mouth open). The observation, discussion, conscious observation of the differences and changes project the visual representation of operations and is a kind of preparation of operational symbols.

Activities like reading of specific images, figures, drawings by properly selected movements (for example, standing up, sitting down, using different hand positions during making sequences), saying verses by syllables (for example, picking up an element by a counting-out rhyme), making sounds (for example, throb, knock, clap or any intoned tune) represent a kind of „counting". For example:

Let ♣ mark a clap, ♥ mark a foot stamp.
"Read" the picture below according to the symbols.

♣♣♣♥♥♣♣♣♥♥♣♣♣♥♥♣♣♣♥♥

Find out the different readings by using movements, sounds.

Counting can be made in different ways in the case of the same picture (number). Many people formulate this in the following way: "a number has different names". This means that the number can be expressed by its decomposed members, by different characters. The purpose of the listed activities is to enable the pupil to make calculations reliably in the learned number ranges, to imprint names, nominations to memory, to recall and use them.

Operations

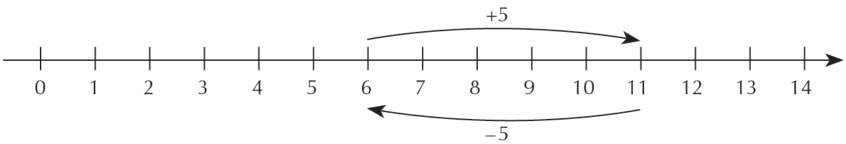
Mathematical operations with whole numbers represent the typical field of appearance and assessment of the phenomenon called additive reasoning in the system of mathematical skills. Literally the word additive refers to an addition, but in the wider sense of the word it also includes knowledge elements of comparing quantities, numerosities. These knowledge elements make us understand that by taking away from a given quantity and by adding the same quantity we get to the initial quantity.

In the process of activities aiming at the formation, deepening of number concept we prepare the mathematical concept of addition (+) and subtraction (–): by reading of the different numbers, sums (for example, 5 walnuts and 2 apples are equal to 3 apples and 4 walnuts) and differences: for example, a picture shows that of 5 boys 1 has not eaten the food that is 4 boys of the 5 have eaten their food. The 5 – 4 is the difference form of 1.

From a content point of view adding (supplementing to a certain amount (for example, 3 + △ = 7)) and partitioning (dividing the whole into two or more parts (for example, 8 = △ + △)) are basically related to addition, but as to their mathematical background they represent the solution of open sentences. Partition allows the production of a number in many different ways, but a number can be produced both by adding and by taking away (for example, number 4 can be produced from 1, 2, 3 by adding and from 5, 6, etc. by taking away). The experiences collected during the various displays, pictorial and text situations typically formulated in words prepare the algorithm of making operations. By the time children can write down the numbers the understanding of operational signs (symbols), their safe use is well founded in the learned range of numbers. In the first two grades we mainly lay the grounds for addition, subtractions and gradually deepen them (in grade 2 extended to number range up to 100), and we develop the need for self-checking.

Outstanding role is given to the interpretation of operations by means of the number line.

For example:



Moving along the line number in two directions connects the operation and its reverse. Arrows showing to the right represent additions, those showing to the left represent subtraction. They illustrate well that 11 is by 5 bigger than 6 and 6 is by 5 smaller than 11.

The conceptual characteristics of multiplication (addition of equal addends), partition into equal parts (for example, by visualization, marking (for example, introduction of 20/4), division (visualization, marking (for example, 20 : 4)), division with remainder (with display, indication of remainder) are prepared through a sequences of activities.

In the course of studying the characteristics of, and relations among operations we mainly make the 1st grade students discover the interchangeability and grouping of the members of addition and look for relations between addition and subtraction. In the 2nd grade we also observe the relations between the changing of addends and the change of the result, the relations between multiplication and division, and we also observe the interchangeability of the multipliers during manipulative activities.

Algebra

The algebraic symbols and procedures composed a special module in the field of Numbers, number systems in the disciplinary division of mathematics. The abstraction needed to the handling of symbols presumes the conversion operation in the Piagetian sense, representing the basic element of mathematical thinking as element of additive and multiplicative reasoning.

Relations, Functions

The subject of relations and functions plays an outstanding role in the development of cognitive abilities. Inductive reasoning (sequences of numbers, number and word analogies) belonging to the Relations, functions topic can be mentioned as an element of multiplicative reasoning. Similarly, the interpretation of proportion as a function also appears during the development of proportional reasoning.

In connection with the development of the counting ability children shall be able to continue declining and increasing sequences of numbers in the set of natural numbers up to hundred. They have to find the rule for sequences where the difference between successive numbers forms a simple arithmetic sequence.

Continue the sequence by adding two members. What is the rule?

1 4 7 10 13 ___ ___

The learners should be able to follow and continue the periodically repeating movements, rhythms. In the case of number sequences they have to recognize if it is a declining or increasing, or periodical sequence.

Continue the sequence by adding two members.

1 3 5 3 1 3 ___ ___

How would you continue the following sequence? Find at least two rules.

2 4 6 ___ ___

The exercises where correlations have to be found between the elements of number sequences, or other sequences (of objects, other elements), or tables also represent the application of multiplicative reasoning. These problems improve both the inductive and deductive reasoning abilities of pupils. It is important to discuss, interpret the many different ways of formulation of rules both from the point of view of development of skills and the assessment of the solutions.

Look at the following sequences of flowers and answer the questions.

a) Draw the next member of the sequence.
b) What rule was used in the preparation of this sequence?
c) If you continued the drawing what do you think the 12th, 15th and 20th members of the sequences would be?

Word problems or parts of them contain ideas the collective discussion of which is educative, thus we should by all means talk about them (for example, the text can be about environment protection, friendship, selfless help, sharing our snack with the fellows, conditions of civilized coexistence, it can be based on family, holiday, geographical, historical, artistic subjects).

Regular dealing with word problems develops accurate, clear and intelligent communication of learners, strengthens the competence of understanding and creating texts, problem solving thinking, creativity, initiating disputes based on reasoning, the need for control and self-control.

By the end of the second grade the students should be able to state the rules of sequences and to continue the sequences to determine the rule based on the difference between the members of the number sequences

Continue the sequence. What can be the rule?

1 3 6 10 15 ___ ___

In the case of most number sequences there is an obvious rule which can be found with the least cognitive effort. One of the elements of inductive reasoning is that the pupil should be able to recognize the "economic" solution from information theoretical point of view, which can thus be called the obvious or the most intelligent solution.

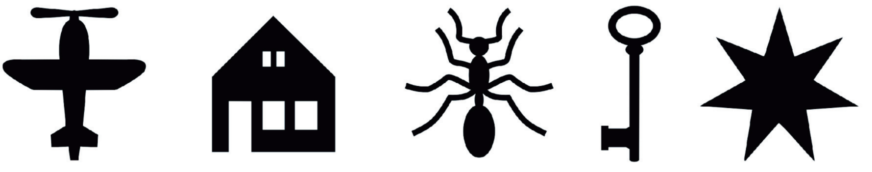
On the other hand it comes from the requirement of developing divergent reasoning that besides improving inductive reasoning all such rules which the learner is able to justify rationally must be accepted as a solution. In the case of the above problem for example the difference between the numbers always increases by one, that is the following member will be by 6 bigger than 15. We also have to accept the simplifying rule-making which does not use the information content of the sequences, but in these cases we have to show during the class that there is "more" in the sequences than for example the following two possible simplifying rules: (1) simple, monotonous sequences where the next member is bigger than the previous one. After formulating this rule we have to accept any two natural numbers which ensure the monotony of the sequences. (2) It often happens among small school children that they consider a sequence of numbers periodical, although this was not the aim of the author of the problem. In this case 15 would be followed by 1 and 3. Thus during the setting of problems we either give a priori the rule of continuation of the sequences (or we should at least refer to the type of the rule to be determined) or rule-making will be inseparable from the continuation of the sequences.

Geometry

Within the system of mathematical abilities, we highlight two components which are closely linked to geometrical contents. One of the actively tested fields of the research on intelligence is spatial reasoning, that is the ability of people to turn plane and spatial forms in mind and to make operations with them like for example rotations interpreted as geometric transformation. On the other hand, proportional reasoning interpreted as part of multiplicative reasoning can be linked to measurement, one of the subsections of geometry. Problems can be set both in the area and volume calculations and in the conversion of units which essentially indicate the developmental level of proportional reasoning. This latter ability is not yet explicitly mentioned in the frameworks fro grades 1-2, in the above we wanted to mention two abilities which are typical for geometrical contents. In this age group the following contents belong to spatial thinking.

The observation of the countless patterns created by transformations (including patterns to be found in the nature, in folk art, in the built environment, in the different human works) prepares the mathematical interpretation of symmetries, repetitions, rhythms, periodicities. The activities promote that *the pupil be able to recognize symmetries, at experimental level (manipulative and pictorial). They should be able to differentiate between the mirror image and the shifted image on the basis of the total view.*

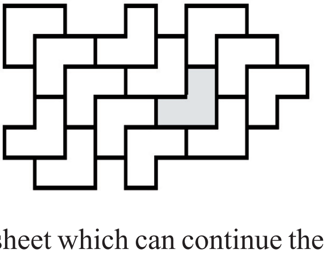
Copy the following illustrations on a transparent paper.



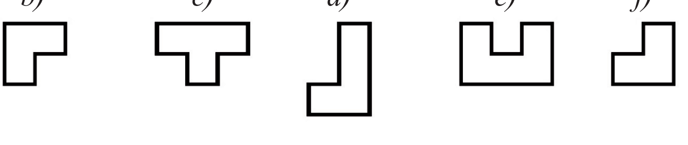
Check which of the illustrations can be folded in a way that the two parts cover each other completely?

Solution: forms 1., 3., 5 can be folded according to the condition.
Typical exercise for testing spatial abilities:

Colour with graphite pencil the sheets which stand in the same way as the grey-coloured sheet.



Circle the letter of the sheet which can continue the above parquet building.
~~Cross out~~ the letter which cannot be used.



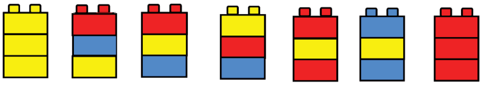
Combinatorics, Probability Calculation, Statistics

Operations of combinative ability can partly be linked to the elements of combinatorics, a content domain of mathematics. By revealing the psychological constructs enabling the mathematical phenomena of permutation, variation and combination we arrive at several other operations (for example, finding all sub-sets of a given set, generation of Cartesian product of sets) which typically do not belong to the combinatorics domain in school mathematics education. Among the mathematical reasoning elements, however, these latter are also manifestations of multiplicative reasoning while from psychological point of view they are part of combinatorial reasoning.

In general, by the end of grade 2 we do not get to the building up of independent system of combinatorial abilities, since this would indicate reasoning in some kind of structure, which in turn requires high level mathematical abstraction skill. Therefore the assessment of different components of combinatorial reasoning is feasible in the case of tasks containing small sets.

In the following the building up of combinatorics is presented through some problems in the foundation stage (grade 1 and 2):

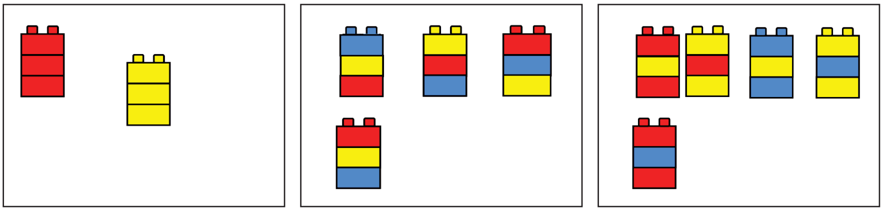
I have built three-level towers of red, yellow and blue Lego elements.
What else could I have built? Draw the other towers.



In this problem the difficulty is in keeping some characteristics constant while others may change. Does a solution fulfill the conditions (three-level, made of red, blue, yellow colours)? Are there any newly built towers which can already be found among the formerly built ones? Assessing children's knowledge it is import to know who and by how many objects extended the set, who were able to create objects different from the existing ones and from those of their mates.

We can make the task more difficult by formulating the problem in a different way:

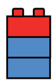
I have built towers of red, yellow and blue Lego elements. Then I arranged them in three groups:



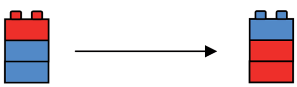
What else could I have built? Put new towers at the correct places.

In the problem above the criteria of systematization is shown by the drawing and not by the text. Finding the criteria is an important element of the problem (one, two or three coloured towers). In this arrangement however, the transparency of the whole system is questioned. It is also a question whether other criteria can be found to the solution.

The second group shows that the elements below each other were created by "reversing" the towers. This strategy works very well here. But it cannot be carried forward to the third group, since here some typical characteristics were left out of the row of problems thus the eventual absence cannot be discovered. It is possible that somebody detects some kind of regularity in the arrangement of elements in the third group, namely that the elements are inverses of each other. In this system however the finding of all the elements cannot be guaranteed, since the drawing does not give an example of the following type:



Thus in the problem different strategies shall be used when finding the one, two or three colour elements. It is possible that for somebody exactly the solution strategy gives the basis of the criteria system and puts the above element into the second group, since

of this tower:  this tower was made by reversion.

By presenting the above problem we wanted to illustrate the diversity of combinatorial reasoning, the direct consequence of which is that in grade 1 and 2 in the assessment process we have to be content if students find some other elements fitting into the given system of criteria.

Detailed Assessment Frameworks of Grades 3-4

Numbers, Operations, Algebra

The correct representation of whole and rational numbers is of key importance in the development of the number concept. There are abilities belonging to additive reasoning which lead to the representation of rational numbers. In people's thinking rational numbers are mental representations of the relations between the numerator and the denominator. With the help of division into parts, we prepare already in the preschool age the empirical basis for learning fractions.

By dividing the whole into equal parts, the notion of unit fraction is developed with the help of different quantities (length, mass, volume, area, angle), then by uniting several unit fractions, fraction numbers with small denominators are produced. During this work the children are performing double direction activities. By cutting, tearing, folding, colouring and fitting the parts they produce the multiple of unit fractions, or they name the produced fraction parts in comparison with the whole. They compare fractions produced from different quantities, put them in order according to their size and look for the equal parts.

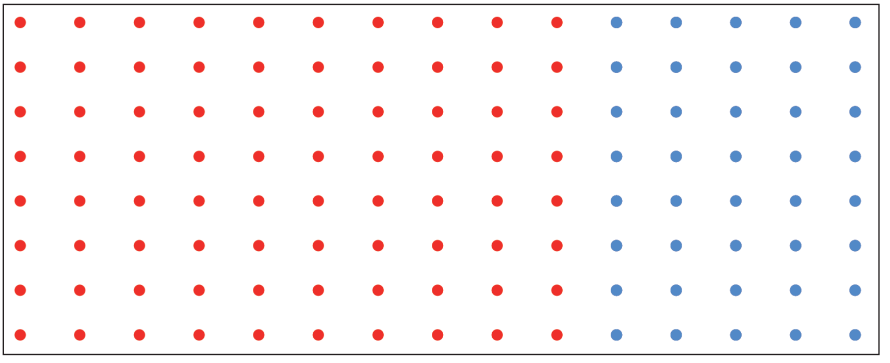
Additive reasoning includes abilities which enable for learning the characteristics of arithmetic operations. The children continuously obtain experiences about the operational characteristics of addition. The computation procedures make possible that the pupils safely give answers to problems which require operations with actual numbers or their comparison.

For example:

The Szabó family made a four day excursion. On the first day they travelled 380 km, on the second day 270 km when they arrived at their destination. On the way back they took the same route. After travelling 400 km they arrived at the night accommodation place. How many kilometers did they have to go on the fourth day?

Working with different object, numbers and word problems offer possibilities for practicing the role of parentheses in connecting into one number and in the multiplication of the sum by members.

For example:
The drawing shows an orchard. The red circles represent apple trees, the blue ones plum trees. How many fruit trees are there in the garden?



The operational properties are consciously used during multiplication in writing.

For example:
Which multiplication is correct?

a)
$$\begin{array}{l} \underline{\;263} \cdot 27 \\ 1841 \\ \underline{\;\;526} \\ 2367 \end{array}$$

b)
$$\begin{array}{l} \underline{\;263} \cdot 27 \\ 1841 \\ \underline{\;\;\;526} \\ 18636 \end{array}$$

c)
$$\begin{array}{l} \underline{\;263} \cdot 27 \\ 1841 \\ \underline{5260} \\ 7101 \end{array}$$

Division as a written algorithm is the most difficult operation. With the help of tools the children learn to divide by one-digit number in grade 4.

During computing operations the different types of control methods which they learn during the acquaintance with the procedure provide safety to the children. Estimation, multiplication, partitioning and the use of pocket calculator are among the methods of checking.

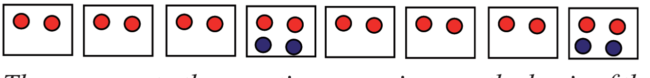
In general, in the fourth grade we offer opportunities for the children to look for different solutions and to compare them. In this way the ability to recognize the existing relations between the models can be developed. The fact that the data of different models are identical, that there is connection between representations and operations are recognized consciously by the children. The teaching of the different ways of solutions and their appreciative use is the guarantee that the children will be able to activate, if necessary modify according to the type of the problem these solution methods in new situations, in case of changed conditions. In this way the knowledge of children can be easily developed. Getting acquainted with, and, comparing different solutions children can judge the usefulness and beauty of different solutions.

Below is an example of solving a problem in several different ways:

The top of a high hill can be reached by a lift. In some lifts two people are travelling at the same time, in others four people. A company of 20 people was carried up by 8 cabins. In how many two and four seat lifts did they travel?

Solution 1: *with activity, using tools*

Children place 8 sheets of paper in front of them, which represent the cabins, they prepare 20 discs, representing the travellers. They put the discs on the papers so that two or four discs were on every sheet.



The answers to the questions are given on the basis of the picture they get: 6 pcs of 2 seats and 2 pcs of four seat cabins were taking the 20 member to the hill.

Solution 2: *trial and error method, using a table*

Number of two seat cabins	1	2	3	4	5	6
Number of four seat cabins	7	6	5	4	3	2
No. of travellers in the two seat cabins	2	4	6	8	10	12
No. of traveller in the four seat cabins	28	24	20	16	12	8
Total number of travellers	30	28	26	24	22	**20**

From this solution more information can be obtained, we get answer for questions which were not formulated by the original problem. For example, how can 30 persons travel up to the hill in eight cabins?

Solution 3: *using open sentence*

Mark the number of used two seat cabins: $\square$

Consequently, the number of used four seat cabins is: $8 - \square$

Number of travellers in the two seat cabins: $\square \cdot 2$

Number of travellers in the four seat cabins: $(8 - \square) \cdot 4$

Total number of travellers: $\square \cdot 2 + (8 - \square) \cdot 4 = 20$

From this it can be determined that the number of two seat cabins is 6.

(Children use the planned trial and error method to produce this result.)

The number of used four seat cabins is 2.

The three completely different ways of solution of the problem above shows that we cannot expect from the children to solve the problems on the basis of only one scheme, we should not insist on following strictly determined steps. That's why it is more preferable to evaluate the selection of the correct model and the solution of the problem within the model.

In these grades we begin the preparation of concepts, procedures which need to be further developed later without making the children consciously aware of what is happening. The organized collection of experiences is only the beginning of a long process (for example, arriving from the fraction to the whole). The mathematical knowledge of the pupils develops in the higher grades in accordance with the curriculum, therefore it is undue to expect children to give precise definition of the concepts they use.

Relations, Functions

In grades 3-4 the pupils can prepare simple graphs and are able to read their data. They are able to look for mathematical models to a given situation with texts, pictures and to match them with data. If necessary they use other mathematical models (sequences, tables, simple drawings, graphs) in the solution of word problems.

Learners can recognize simple correlations, express them by examples, basic generalizations. The relations can be recognized, correlations can be read from figures, tables.

In these grades, the acquired knowledge, skills and abilities can be evaluated by means of tasks formulated by simple instructions. Here we mainly ask the pupil to perform an acquired, practiced step or sequences of steps.

It may happen that we do not use mathematical symbols for the description of the problem, but rather drawings, figures and we often expect from the children the steps to be made not in “mathematical” form, but in drawing or by some kind of illustration what’s more, in the everyday life we can expect some kind of activity. Through some examples, we present below how many different types of problems can help to practice inductive rule generation and to follow the rules.

Continue the drawing in a way it has been started:

¤ *Δ* *Δ* ♥ # ¤ *Δ* *Δ* ♥ # ¤ *Δ* …………………

Add the missing numbers in the “number snake”.

Continue the sequences below with 3 elements on the basis of the given rule: the difference between the elements always increases by the same.
1 3 6 ………………………………………………………….
Look for a rule yourself and continue the sequences on this basis.

What symbol can be found in the square marked by (5;C)? …….

D		☺		⌂		⌂	
C	⌂		☼		☼		☼
B		☼		⌂		☼	
A	☼		☺		☺		⌂
	1	*2*	*3*	*4*	*5*	*6*	*7*

Colour the quadratic lattice below according to the following instruction.
yellow: (3;f) (4;e) (4;g) (5;g)
red: (2;f) (3;e) (3;g) (4;h) (5;e) (5;g) (6;f)
green: (3;c) (4;b) (4;c) (4;d) (5;c)
brown: (1;a) (2;a) (3;a) (4;a) (5;a) (6;a)

h						
g						
f						
e						
d						
c						
b						
a						
	1	*2*	*3*	*4*	*5*	*6*

What regularity do you find among the index numbers of brown squares?

In case of proportionality there are many possibile ways for selecting a problem. Every conversion of measurement unit, buying, uniform motion, work, enlargement, etc. are eligible for the formulation of simple routine problems.

How much does 6 kg potatoes cost if 4 kg costs 312 HUF?

Zsófi travelled the 27 km long bicycle route in one and half hour with constant speed. How far did she get in 10 minutes?

Children measure the length of their classroom by steps. Csaba could take 18 steps from one wall to the other, but Julcsi takes 24. Who has the larger step?

Grandma prepared pastries for the children, in all 32 pcs. She made croissants and pretzels. How many of each?

	*5*	*6*	*7*	*10*					
	*27*								

On Monday Zoli received a piggy-bank and 200 Forint. He put the money into the piggy-bank, plus he put a fivo-forint coin and a tan-forint coin into it every evening. On which day did he have 320 HUF in the piggy-bank?

Among the word problems the ones describing the events of real life, some kind of motion, changes are of great importance. We most often describe change of temperature, growth, movement. The pupils have to recognize these changes, sometimes illustrate them, and look for relations, correlations, and regularities. The following sequences of problems illustrate the very varied possibilities of recognition of relations and of following the rules.

When Panni was born her mother was 25 years old. How old is her mother today if Panni is 9 years old? How old will be Panni, when her mother is 50? When will the two be 99 years old together? Make a chart about the age of the two persons and based on the chart formulate other statements.

The distance between two cities is 190 km. Trains depart from each city every morning at 8 o'clock towards the other. One of the trains takes 50 km in an hour, the other 45 km. Make a drawing about their movement and find out when they will meet.

In a reservoir there is 4800 hl of water. A pump is lifting out 8 hl water per minute and 2 hl water is added to the tank via a pipeline system. When will the tank become empty?

Péter is making a puzzle. He has to make drawing according to an instruction starting from a certain point of a squared paper. The arrows show the direction, the numbers show the number of steps. What did Peter draw if he followed the instruction correctly?

8↑ 5→ 2↓ 3← 1↓ 2→ 2↓ 2← 3↓ 2←

Geometry

Through the knowledge elements of spatial abilities the learners will be able to create line patterns, spread patterns, parquet patterns, colouring, drawing with templates or on net.

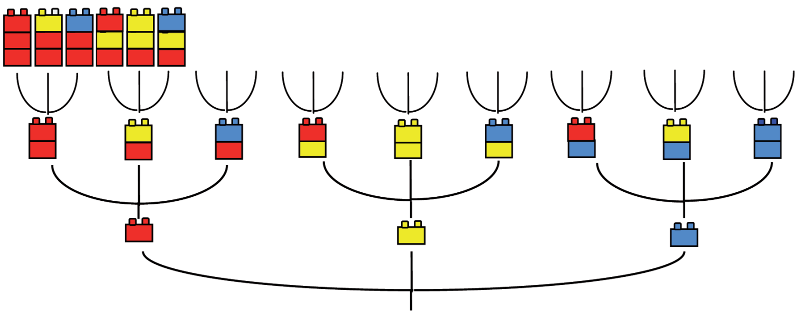
In the field of measurement there appears the requirement for the conversion of measurement units. The pupils should only know the conversion of units in cases to which – in principle – they can connect realistic experiences. Thus the technique (and together with this the safety) of mechanical computation can be taken over by proportional reasoning rooted in real life experiences.

Combinatorics, Probability Calculation, Statistics

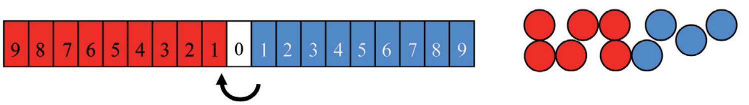
In these grades priority is given to the development of systematization skills in the combinatorics and probability content domains. For example, during the lessons we can give the task to the children to build three level towers and to try to build towers as varied as possible. They should look for all the options. During the lesson the teacher asks the children to observe and collect all the ideas based on which they can say: all the possible towers were made. The pursuit of completeness does not necessarily develop in the children by itself not even after a longer time. From the side of the teacher problem proposal or giving support may also be required: Is there any other, or there are only so many and no more? How can a small child realize whether he/she was able to find all possible options and if not what is missing? An important and good opportunity for this is that they somehow arrange in front of them the built up towers in a “beautiful” way.

Some of them pay attention to the colour of the lowest element of the tower and put aside those which they started to build in red, they separate the blue and yellow base towers. In this case they may realize that the same number of towers should be in all the three groups and this can be a starting point to the determination of shortage, perhaps to the finding of the missing building. It is commonly said it is because of “symmetry” that the same types of towers can be found in all the three groups. This concept means that there is no explanation why the building can be continued in many different ways if we put one colour below or if we put another one.

The advantage of this arrangement is that it can be continued: whatever the starting colour was, three different colours can be put in the middle and whatever the first two were the building can always be finished by the third elements in three different colours. This type of system building can be illustrated by a diagram looking like a tree (this is how it is called: "tree-diagram"):

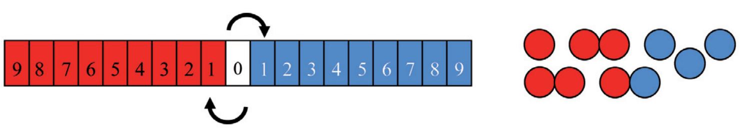


In the course of the improvement of probabilistic reasoning a lot of different games are practiced, for example, games with discs. The game is played in pairs. The members of the pairs select a side on the table and move a figure starting from 0 (white field). They can step one to the right if after throwing up 10 discs there are more red than blue discs and they can move one to the left is there are more blue discs than red ones. (If the number of the discs is equal they do not step.)



In our example one step to the right is allowed. The winner is the player on the side of whom the figure is standing let's say after 20 throws. (If it actually is at position 0, it is a tie). The game is simple and the probability sense suggests that the blue side is as good a choice as the red side. When they compare their experiences on class level, they will find the same.

On another occasion the children play with two figures and 10 discs so that "A" can move if the number of red discs is even, while "B" will step if the number of blue discs is even. Both players move their own figures.



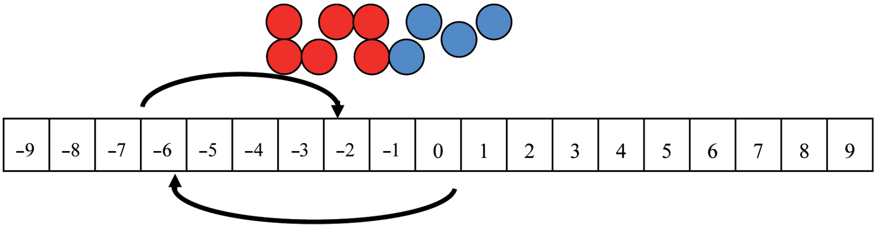
In our example both players step one. They have to complete a number of games so that they could make the following conclusion: the game will always be a draw, since either both players can step or neither of them. With this problem it is worth however to "make a joke" with the children, since in this way they can acquire the idea that 10 can only be divided into sums both members of which are either even or odd.

If we change the number of discs to 9 now, we again play a game where the probabilities are the same.

Further observations can be made by the generalization of the problem. For example they play with different even or odd number of discs. When developing students' reasoning, it is much more motivating for the children to obtain experiences about the division of sums into even or odd numbers by playing a game, than by doing mechanical operations.

For another didactic purpose, the pairs again play with 10 discs. One figure starts from 0, but here the table is replaced by a number line. After throwing the discs the players shall make the same number of steps to negative direction as the number of red discs dropped on the table and to positive direction according to the number of blue discs.

For example I threw this:



I step six to the negative direction and then from the arrival point I step four to the positive direction. I could have first stepped four into the blue direction and then six in the red direction. (Shall I finally arrive at the same place? Is commutativity working in the case of the negative numbers too?)

Now they throw ten times in a row in a way that the figure always steps further from the point where it stopped after the previous throw. Before starting the game the children should make a guess where the figure will most often arrive at from the following options after 10 throws: –6, –3, –1, 1, 3, 8. Shall it arrive at point 8? Or at point –3? Before starting the game everything is possible. Our idea about probability suggests that the many throws somehow compensate each other and the guess should be somewhere around 0. Yes, but now 0 is not among the possible guesses, thus 1 or –1 or perhaps 3 or –3 can also be good.

After some games the teacher asks the children where the different pairs arrived, for example the following notes can be made: –2, –8, –2,– 4, 0, 0, 6, 6, 4, 8, 2, 2

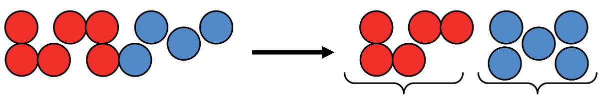
Can it be *accidental* that all of them arrived at even number?

A new round may confirm the guess and the search for explanations can be started.

We can collect the possible throws and the possibilities for the length of a step:

10 r = –10	10 b = 10
9 r + 1 b =–8	9 b + 1 r = 8
8 r + 2 b = –6	8 b + 2 r = 6
7 r + 3 b = –4	7 b + 3 r = 4
6 r + 4 b = –2	6 b + 4 r = 2
5 r + 5 b = 0	

Or they simple look at what is happening if one blue disc is changed to red:



Another statement which the children can discover themselves and can feel much closer than by simply getting the teacher's word: "If I reduce the minuend by one and increase the subtrahend by one the difference will be reduced by two".

Thus whatever we throw with 10 discs we will always arrive at an even point after the first throw. And during the further throws we will always step even numbers. During these steps the children get experiences about activities required to the interpretation of the opposites of positive numbers, about the addition of positive and negative numbers and about the fact that the relation about the parity of the sum will remain valid in the circle of negative numbers, too. Children can get more realistic experiences about an *impossible event* with regard to probability concept, than by getting such an extremely obvious example that the sum of numbers thrown by two cubes can never be 13.

Detailed Assessment Frameworks of Grades 5-6

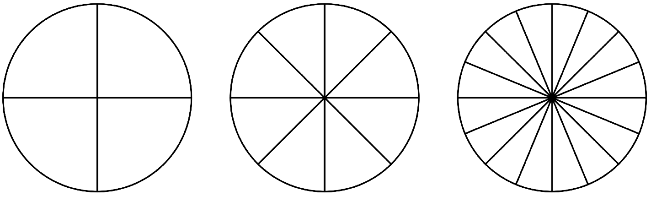
Numbers, Operations, Algebra

In grades 5-6 whole numbers (both positive and negative) up to arbitrarily high absolute values turn up in the school, that is together with keeping the empirical basis of numerosities typical of the earlier grades the representation of "big" numbers should also be developed. From a mathematical point of view the device of this is the normal form of numbers, from a psychological point of view the element of additive reasoning. In the comparison of the size of the numbers the interchangeability of relations "smaller than" and "bigger than" appears as elements of additive reasoning.

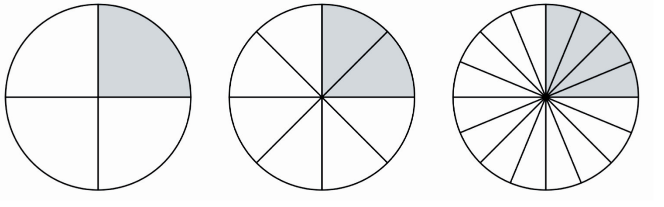
In the sphere of numbers connectable to the empirical basis the varied and purposeful forms of activities will certainly continue in grades 5-6, too: working with objects, cutting, decomposition, making, filling in of place value tables, reading numbers from them, writing down verbally pronounced numbers, representations, reading numerals, comparisons on number line, etc. The diversified experiences help for example the deepening of the concept of fraction, decimal number, negative number, the varied representation of the same values (for example with additions, simplifications) and the representation of the same values in different forms (for example decimal number form of a fraction and vice versa). Only the concepts and contents which were experienced in many different ways will be long lasting, easily usable, and can be recalled.

In the case of the fractions it is important to show (with a lot of folding, cutting, putting out of same cubes, using varied units, by drawing, etc.) that we can divide a unit into equal parts in many different ways, thus a given fraction value can be represented in many different ways.

On the figure below we have divided three circles of the same diameter to 4, 8, 16 equal sectors. Colour one quarter of the circles.

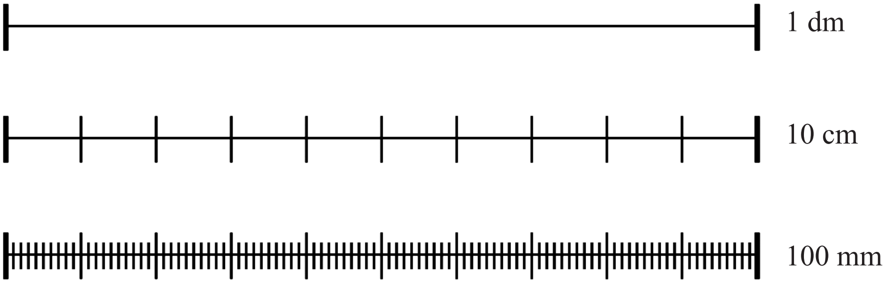


Solution:



The figure clearly shows that 1/4=2/8=4/16. Should these identical circles represent alike cakes, the child eating 1/4 cake would eat the same quantity as children eating 2/8 or 4/16. The only difference is that one of them would get 1, the other 2 equal, but smaller pieces, while the third one 4 equal, but even smaller pieces of cakes.

Mark one third of each of the three line segments. Describe the received quantity in terms of the unit indicated at the end of the line segment. Compare the quantities.



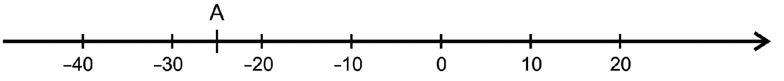
Solution: *By copying them to a transparent paper, with folding we can also see that 1/5 decimeter is exactly 2 cm (2/10 decimeter) and exactly 20 millimeter (20/100 decimeter), thus it is true that 1/5=2/10=20/100.*

Making a lot of similar tasks will give the basis to the understanding of the extension and simplification of fractions and will explain why changes during application are necessary (looking for common denominator).

The representation of wholes and fractions on the number line illustrates well the understanding of the numbers' relations with each other, their increasing and decreasing order.

Answering questions related to the number line deepens the understanding of the number concept and the concept of operations, too.

Answer the following questions.



Which number is smaller 20 or –40?
Which number belongs to point A on the number line?
What is the distance between –10 and 10?
Put the absolute value of the numbers 1,5; –17,8; 0; 65; –197 in increasing order.
Put the numbers –325; 3,25; 32,5; 0 and 0,325 in increasing order numbers.

The pupils should become capable of representing the learned numbers on the number line, to determine precisely or approximately the number belonging to a given point on the number line or to compare the numbers according to their size.

In addition to performing the verbal and written operations in the appropriate order with the correct results we also make efforts in the first two grades of the upper grades that the children learn methods, procedures making computations simpler, faster (for example, by using operational properties, parentheses). This also confirms the deepening of concepts, the increasing of awareness of operational algorithms.

By the end of grade 6 the pupils get acquainted with the basic operations in the set of rational numbers.

We only allow the use of pocket calculators during the lessons if the children possess the basic computation algorithms and are able to give adequately correct estimation of the final result. We generally do not allow the use of calculator in the paper-pencil tests. One of the main reasons for this is that by this we provide unequal technical conditions (plus the problem of the use of technical tools which look like a calculator but have much more sophisticated functions).

The pocket calculators with different "knowledge" serve the interests of our learners if these tools do not take over too early the steps, operational elements needed to the development of students' reasoning. The problem solution model is born in head, the calculator can be a tool of implementation. For example, when we teach how to solve equations, children are working in head and in writing, since we want to make them understand and to teach them the algorithm of solution. In the case of more difficult word problems the challenge is the setting up of the mathematical model and the calculator, or its equation solving program can perhaps be used if the model already exists. If for example we would like to check the correctness of an estimated or computed result by fast replacement, the use of calculator can also be justified. Knowing the actual conditions we can make a good decision about when and why we let the children use the calculators, computers. The use or the refusal of use should always be supported by rational pedagogical reasoning.

The application of highly developed information technological environment requires the improvement of good estimation skill. If for technical reasons the machines are not working, the good estimation skill gives a kind of security (for example, in the calculation of the amount to be paid /or to be claimed back).

New elements of understanding word problems

In grades 5-6 the continuously growing knowledge (operations covering the rational numbers, order of operations, knowledge of proportions (direct and inverse) and calculation of percentage) make possible the introduction of more complex word problems. The more demanding implementation of solutions (writing down, aesthetical aspects) is formulated as a requirement, it is realized that the rounding rules can be overwritten by real life (for example, if we need 56,3 m of a wire fence which can be bought in meters, we have to buy 57 meter, if based on the actual calculation of the surface we need 37,2 pcs of tiles to covering, we buy minimum 38 pcs and some additional), the estimation skill and the need for checking, self-checking is developing.

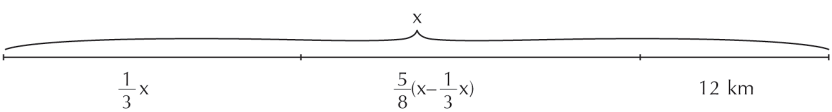
In these two grades the word problems mainly serve the development mathematical reasoning (for example, solution of simple first-order equations and inequalities by means of deductive reasoning processes), the improvement of proportional thinking (for example, conversion of standard units, direct and inverse proportions, simpler percentage calculation tasks), of problem solving skill (recognition of problem, identification of problem and solution) and the development of knowledgeable-analyzing reading.

During the development the consecutive steps of solving word problems are continuously recognized by the pupils (good understanding, interpretation of the text, clear separation of the conditions and the question, recognition of data (including the unnecessary data, too), recognition, stating, displaying, writing down of relations, links read from the text, preparation of solution plan(s), putting down the estimation of the result, calculation of the result (with written and verbal operations), its determination, checking, comparison with the estimated value and real life, preparation of an answer in words), the need for searching for different solutions is developing.

The pupil has to be able to solve simple equations with optionally selected method, to solve simpler word problems by deduction, proportional problems, to represent the solutions on number line. Of the solution methods mention should be made - besides deductions - of the methods using drawing, figures, segments, number line. In many cases these drawings, figures show if the learner understood the problem, the task. Some kind of actual representation of texts by drawings, figures can give a lot of information to the teacher about the current level of the slowly developing abstract reasoning of the pupil.

Edit and Dani went on an excursion. On the first day they made one third of the planned route, on the second day 5/8 of the remaining distance, thus they had to walk 12 km on the third day in order to get to the destination. How long was the route of the whole tour?

Solution in segments*: x marks the length of the whole tour.*



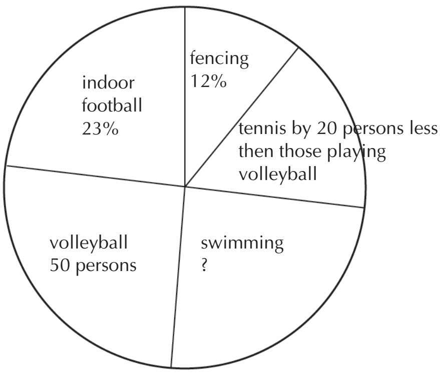
12 km is 3/8 part of the 2/3 of the total route
4 km is 1/8 part of the 2/3 of the total route
8 · 4 km = 32 km is 2/3 of the whole route
Length of the whole route: (16 + 32) = 48 km
Checking can be made by the calculation of the parts and by their summing.

Find a connection between the following quantities.
a) The price and the height of the Christmas tree
b) Travel time and speed of the car (let the route length be 20 kilometer)
c) Number of slices of a birthday cake and the size of the slices (we cut equal slices)
d) Quantity and price of green peas
e) Side and perimeter of a square
f) Price of the ice cream and number of balls
Solution: *Discovery, formulation of the correct correlations between quantities.*

Answers which can be expected from pupils can be for example:
a) In the case of the same type of Christmas tree we pay more for the taller tree, than for the shorter.
b) If a car goes twice as fast then it will take half the time to cover the 20 km.
c) The more equal slices I cut of the cake, the smaller the slices will be.
d) The price paid for the green peas changes in direct proportion with its quantity.
e) The lateral face and perimeter of the square change in direct proportion.
f) The number of ice cream balls and its price change proportionally.

48% of the monthly family income goes for the payment of different credits, invoices. In this month the family covers its living (food, clothing, repairs, entertainment, etc.) from the remaining 104 thousand Forint. How much is the family income in this month?
Solution: *The remaining money (100 – 48)% that is 104 thousand Forint is 52% of the monthly family income.*
1% of the family income is 2 thousand Forint, thus the total income is 100×2 thousand Forint, that is 200 thousand Forint.
Checking of the problem: 48% of 200 thousand Forint is 96 thousand Forint, this together with the 104 thousand Forint is exactly 200 thousand Forint.

200 sportswomen and sportsmen disclosed which their favourite sport wss. We will show this on the diagram below. What percentage of them selected swimming as the favourite sport?



Solution:	*100%*	*200 sportsmen*
	1%	*2 sportsmen*
	23%	*46 sportsmen (indoor football)*
	12%	*24 sportsmen (fencing)*
		50 (volleyball players)
		30 (tennis players)

Totally: 46+24+50+30=150 sportsmen
Swimming is the favourite sport of 200–150=50 sportsmen
50 is exactly the quarter of 200 that is 25%.
Swimming is the favourite sport of 25% of the interviewed sportsmen.
Checking can be made for example by adding the partial sums.

Requirements of constructing text for word problems

At the beginning of the upper grades the extended mathematical knowledge contributes to the description of mathematical models by symbols. In spite of this even at these grades there is still a need for reading texts, information, instruction, questions from activities, working with objects, pictures, figures, drawings. If the texts constructed by the students for the computation problems, open sentences are faulty, it is worth to show mathematical expression matching well to the problematic text and compare it with the initially given mathematical model. The presentation of differences, deviations helps the pupil to understand where he made a mistake. If somebody cannot (does not dare) to start the formulation of a text to a model, the teacher should begin it encouraging the learner to continue and finish the text. If this does not help the teacher should tell several simple adequate texts so that the learner understands clearly what his/her task is.

As a result of appropriate development, children become able to create more and more complex and better formulated texts to a given mathematical model. In general the texts relate to the applications within mathematics, to the everyday real life surrounding the children, but we should direct the attention to texts relating to the natural sciences, too. Models produced by using special correlations (formulas) taken from this field (for example, relations between route-time-speed, measurement data, use of graphs) give good basis for the implementation.

Nora had 1200 Forint. She spent 3/5 of it. Put questions to the text.

Solution: *a) How many did Nora spend?*
b) How much money was left to her?
c) What portion of the 1200 Forint was left?
d) What percentage of the money did she spend?
etc.

Create a text to the following computation problem.
2(300+100) = 800

Solution (for example): *I had 300 Forint saved, I received an additional 100 Forint from my grandpa. My father doubled my money for my birthday. How many Forints do I have?*

Create text to the following open sentence.
2(1kg + 3kg) = x kg

Solution: *Kati was sent to the shop twice by her mother and both time she had to buy 1 kg sugar and 3 kg of potatoes. How many kg of food did she take home after the two shopping trips?*

Create text to the following open sentence.
2 (30 + x) = 200

Solution: *One side of a land of rectangular shape is 30 m, its perimeter is 200 m. What is the size of the other side?*

Create word problem to the following relation.
a×b = 50, (a and b are positive whole numbers)

Solution: *The area of a rectangle is 50 units. What are the sizes of its sides?*

It is advisable to make the children calculate the length of the lateral faces, since there are several possible solutions here. 50 is divided into the product of two factors in all possible ways: 1×50; 2×25; 5×10. By interchanging the factors we do not get a solution different from the above, a new rectangular. Thus the lateral faces are 1 unit and 50 units long, or 2 units and 25 units long, or 5 units and 10 units long.

Relations, Functions

Relying on the solution of previously solved tasks on proportional reasoning, students learn the concept, definition of direct proportion. They will be able to recognize direct proportions in the practical problems, and also during learning science topics in the school. They can solve with certainty simple proportional problems of everyday life by means of deductive implications.

During the studying of relations between variables the learners gain experiences about the recognizing of inverse proportionality, about the determination of their matched value pairs.

The proportional implications improve the perception of correlations of the learners, their abilities for making conclusions. The learner will be able to recognize relations, correlations in simple examples. In the case of the simplest linear correlations which occurred often before children are able to add the missing elements, to present the data in tables. They have to meet with non-linear relations, too, what's more it is advisable to check the same thing from several points of view.

In this age phase of the development of inductive reasoning the learners are able to determine the missing elements, or in case of known elements to formulate the rule. They can continue a sequence according to a given rule and to induce a rule from some elements. They can also describe the recognized rule by a formula.

In this school phase students' location determination skill is improving. They are able to find points according to the given properties on a number line, to represent number intervals, to demonstrate data described by terms like smaller, bigger, at least, maximum, or to read from a figure. They know the Cartesian coordinate system and the related terms (axes, origin, index, coordinates, and quadrant). They can represent given points in the coordinate system and read coordinates of points.

They are able to prepare diagrams to relations given in tables and to give the table elements on the basis of the diagram. They recognize the linear function and can represent it on the basis of its points. They can recognize, write down, and represent relations in the simple examples taken from everyday life.

They can solve simple percentage calculation problems using direct proportionality, proportional deduction (for example, shopping, savings, agenda). In the course of practicing these tasks in parallel with the discovery and use of the necessary algorithms they learn the basic terms of percentage calculation: basis, interest rate.

Initially the problems formulated by mathematical symbols can be used for the presentation of the acquired knowledge, skills, and abilities. Through them the mathematical structure of the problem is transmitted without any "disturbing factor", in most cases we refer to the operations, algorithms which should be used during solution, and in many cases mathematical symbols can be found in the text of the problem.

Calculate 15% of 120.

Prepare a number line with corresponding scale. Indicate numbers with the following properties. $-3 \le x < 9$ *and x whole number*

Indicate in the coordinate system the points A(–2;1), B(3;1), C(4;3) and D(–1;3). Connect them in alphabetical order. What is the name of the produced plane figure?

Draw points in the coordinate system the second index number of which is bigger than the first.

What is the connection between the data of the following table?

Time passed (hour)	*1*	*2*	*3*	*4*
Route travelled (km)	*4*	*8*	*12*	*16*

Find a rule to the data of the following table. Based on the rule add the missing data.

x	*8*	*4*	*2*		*0*
y	*4*	*8*		*1*	

The last three tasks is an example that learners can be asked to solve problems on this very simple level of application, where several correct answers, solutions can be given. By giving these types of tasks we can prepare the studying of more complex, problem-type, authentic tasks. Certainly, this aspect can only appear in the course of teaching, during assessment one has to refer to the possibility of several solutions.

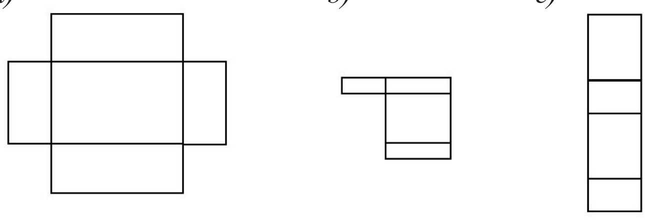
Geometry

In addition to the two abilities (spatial and proportional) playing a role in the former grades, due to the concept enrichment in grades 5-6 it is possible to create several different tasks to the geometrical contents which allow the diagnosis of the development level of inductive, deductive and systematizing abilities.

Typical problem for the testing of spatial skills:

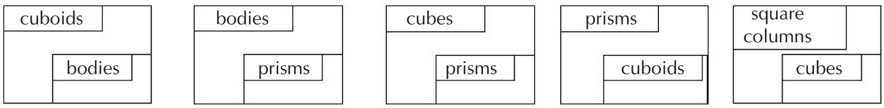
Add the following figures so that all of them be a net of a cuboid.

a) *b)* *c)*



Example of a problem which evaluates the systematizing skill on a geometrical content:

Are the nominations written in the figures below on the right place? **Circle** the letter mark where they are on the correct place and ~~**cross out**~~ where they are not.



a) *b)* *c)* *d)* *e)*

Finally, we present an example, where several different mathematical abilities can be used during the solitions, thus for example the elements of deductive and combinatorial abilities.

The rounded value of the volume of three same size bottles is 2 liters. The value of the volume of one bottle given in dl is whole number. **Answer** the following questions.

a) At most how many deciliter could the total volume of the three bottles be?
b) At least how many deciliter could the total volume of the three bottles be?
c) At most how many deciliters could the colume of one bottle be?
d) At least how many deciliter could the volume of one bottle be?
e) **Give** all possible volumes of a bottle in dl.

Combinatorics, Probability Calculation, Statistics

In the field of combinatorics, probability calculation and statistics the development of basic skills and the deepening of content knowledge of the subject are relevant objectives in this age group, too. In addition to the possibility of developing combinative and correlative abilities embedded in the content there will be an opportunity for the mathematically correct foundation of data handling and data presentation and of the probability event based on theory of sets. In the system of mathematical reasoning the ability for correlative reasoning can be interpreted as a form of multiplicative reasoning. Here the recognition of relations between data sequences and the formulation of the problem is the task where the correlation is not only not linear, but in general cannot be described by a simple formula (even so in many cases the relation is not deterministic). In the world of mathematical phenomena the development and assessment fields of correlative reasoning belong to the world of the statistical phenomena. The formulation of correlative relations like for example, „The more vertices a polygon has, the more diagonals it has" or „the third power of bigger numbers is also bigger" can be regarded less valuable. Thus the correlative reasoning can primarily be improved by experiencing statistical phenomena.

Diagnostic Assessment of the Application of Mathematical Knowledge

Detailed Assessment Frameworks of Grades 1-2

Numbers, Operations, Algebra

In the lower school age groups the word problems have dual functions. On the one hand they are used for mastering arithmetic operations, on the other they develop the problem-solving skills. In both cases it is typical that the text emulates the experiences of everyday life and the cases of children's world of fantasy making possible for the children to imagine or to model the story. At the beginning we cannot expect in grades 1-2 the conscious use of the solution steps of word problems, the teacher's help is needed by giving honts, formulating simple questions.

In the early phase the word problems describe activities, stories the playing or imitations of which lead to the solution. The problems became realistic when the everyday observations, visual and other images stored in the memory get an active role in the solution of the problem and the learner creates a mathematical model by using them during the solution of the tasks.

Look at the picture below carefully and tell a short tale, story about it. Also make up number problems about the picture.

The guideline to the solution of these types of problems usually contains the identification of mathematical terms and symbols, nevertheless the correct model creation reconcilable with real experiences will be decisive.

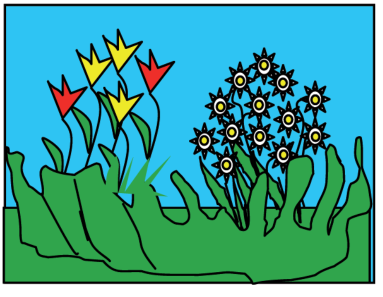
It is clear that the same problem can be a routine word problem in upper grades and can be regarded a realistic problem in lower grades. Most probably the following example belongs to the realistic category for the majority of learners of grades 1-2, while it is a simple routine task for the learners of upper grades.

Every child gets three plums after lunch. How many plums will be put on the table if 6 children are having lunch?

Six pupils in the class play the children sitting at the dinner table. Every child gets 3 plums. Children will determine how many plums they have got together.

It is easier to interpret a text if it is about a specific picture or situation. The text formulated about a picture can be an example of the inverse direction activities, where the task of the child is to make a picture to the text. The problems can be made realistic by making the children describe – in connection with pictures – their experiences, create questions which can be answered on the basis of the picture.

For example

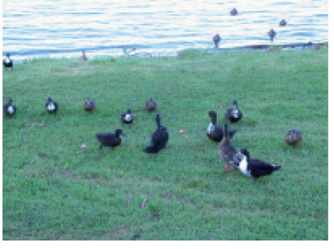


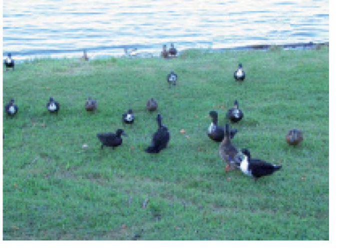
In the garden tulips and white daffodils are flowering. How many tulips are flowering if 2 tulips are red and 3 tulips are yellow?
How many more daffodils are there than tulips?

It is good if the translation of the word problems into number problems or open sentences is preceded by the representation of picture pairs showing the changes well. The reading about the picture pair, the connection of the text and the picture pair shows the recognition of the relationship between the given and the missing data. Picture pairs recalling real situations make possible the creation of real problems.

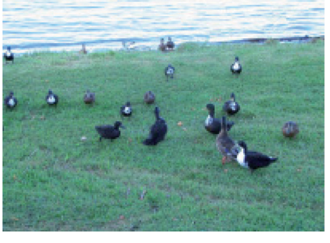
For example:

Describel what happened between the two shots if the photos were taken in the order shown.





What happened in the reverse order?



Word problems given by telling a story become realistic for the children, if they can be represented by manipulation with objects or by drawing. At first the tools and drawings are realistic, they show what the story is about. Later we can expect from the children the interpretation of simpler drawings, more abstract figures. This process at the same time show how an authentic task provoking activities turns into a routine word problem during the development.

Mother sewed 6 buttons on Évi's coat, 2 less than on Peti's coat. How many buttons were needed on the two coats combined?

Level 1: Putting real buttons on the drawing of two coats.
Level 2: Instead of buttons, putting of discs under the children's names.
Level 3: Drawing circles or dots corresponding to the number of buttons after the initial of the children's name.

Other examples of realistic problems building on the children's experiences:

All of us will put on gloves for the walk today. How many pairs do we have to prepare if 5 boys and 4 girls are going out?

Discussion of the terms contained in example (all, pair, 5, boy, 4, girl) contributes to the preparation of the mathematical model.

How many nights do we sleep from Monday morning till Sunday evening?

A lot of significantly different mental models can be prepared to this problem, including the mental number line, the drawing of calendar.

The children will be able to formulate questions and to create problems on the basis of examples of word problems interpreted and solved by activity.

Tomi has 15 toy cars. His younger brother, Dani has 7.
Ask questions.

Children can make several questions.
– *How many cars do the two children have altogether?*
– *How many more cars does Tomi have than Dani?*
– *How many more cars does Dani have to collect to have the same number of cars as Tomi?*
– *How many cars should Tomi give to Dani so that the brothers have the same number of cars?*

The above activities prepare the connection of word problems to mathematical models. First the expression by numbers, symbols and operations of the relations formulated in words is made by collective activity. The collective model creation can be followed by independent activity, where we expect the connection of the simple word problem to the number problem or to the open sentence.

For example:

Which open sentence matches the text? Connect the open sentence corresponding to the problem.

Marci went fishing to the lake.
He threw back 8 of the caught fish.
He returned home with 5 fish.
How many fish did Marci catch?

$8 + 5 = \square$

$8 - 5 = \square$

$\square - 8 = 5$

$\square - 5 = 8$

$\square + 5 = 8$

Open sentences 1, 3 and 4 are all rational models of the word problem.

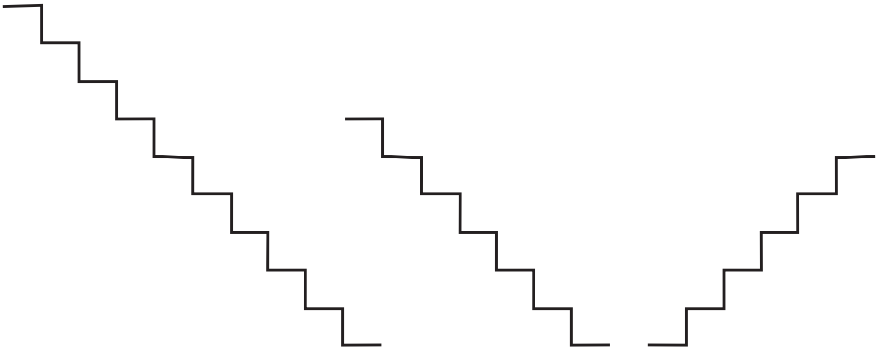
The mutual relations between the texts of examples and the determination of the operation needed to the solution are promoted by problems which require the pairing of text and number problem or open sentence and contain a mathematical model which does not fit to any of the word problems. In this case we can ask for making text to the number problem or open sentence. We can expect and require that the verbally formulated word problems contain real data, connect to the everyday life or real experiences of the children.

The above activities prepare the recognition of solution steps of the word problems. The appropriately gathered and written down data collected from the information of word problems formulated in colloquial language, the description of the relations between them or their representation by activities, the correct estimation of the answer to be given to the question indicate the mathematical model leading to the solution. The creation of the model is the most difficult step of the problem solution. The solution within the model is followed by connecting the solution to the original problem. The children, by comparing the found solution with the text data, with the preliminary estimation and reality evaluate the reality of the solution, too.

In the first years of schooling, children get acquainted with numbers in the course of real problem settings. They make observations, comparisons and measurements. They recognize the sensible properties of objects, persons, things, and select them based on their common and different characteristics. During their activities they gain experiences about the properties, relations of the numbers.

For example, they become able to find solution to the following problem by evoking their experiences about walking on steps:

Which staircase could you walk through in a way that you always skip one stair? Circle the number of stairs, which can be stepped on this way, and ~~cross~~ the number which not.



9 *6* *5*

The presentation of authentic problems creates real, lifelike problem situations for the learners. In the course of this they process problems about which they can have personal, real experiences. We can also present new situations which are regarded by the children authentic based on the stories heard from others. In many cases the problems – as in real life – have several possible solutions. The solution depends on the conditions influencing the event and on the conditions which are prevailing in the given situation. In early school age we cannot expect from the children the taking into account of all the conditions and the recognition of the possible situations. We will be satisfied with the presentation of a possible solution to the problem.

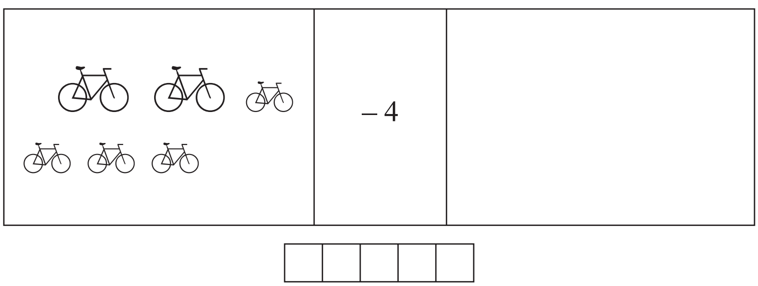
Marci and his younger sister Zsófi go to bed at 8 o'clock in the evening. They have to get up at 6 o'clock in the morning, since the school is far from their home. How many hours can the children sleep?

We can help the solution of the task by showing a clock which strikes every hour. Set the clock to 8 o'clock and the children should close their eyes. While time is passing (now speeded up) the clock is moving. Children can open their eyes when the clock shows 6 o'clock. The teacher makes a clang at even time intervals. During the game, by the speeding up the time, children experience in this new situation what happens to them every day. They see the example, how long lasting events can be played and made repeated several times. Based on their experiences they can state that the characters of the story can sleep maximum 10 hours.

Greater imagination is needed in cases when for the illustration of the problem we use objects which are not real, but still touchable, movable symbolic objects. It is important that these objects be first selected by the children or perhaps the teacher should offer different options. Since one of the main features of authenticity is that the simulation of a problem situation *realistic for the learner* can be made by the definition and solution of the problem.

The next step after making illustrations by objects can be the illustration of problems by pictures, drawings. At first we can connect word problems to photos of personal experiences. Based on the photos the children recall the real events, formulate their experiences, tell what they have lived through, and talk about their observations. Based on their memories they can supplement by data the story told by the teacher, or can put questions themselves. These conversations can contribute to their being able to make stories about photos on their own later.

For example:
Prepare the second picture. Describe what could happen. Describe it in arithmetical language.



This picture can recall the experiences of the children who regularly go for cycling with their parents, brothers and sisters. If they do not have bicycle their wording may express their wishes. They perhaps have seen cyclists on the streets or visited a bicycle shop. Their experiences collected from real life may have an effect on their stories.

For example, they can tell stories like: In a six member family everybody has a bike. On the week-end four of them went for a bicycle tour. How many bikes were left at home?

The problem solution can be made easier if it is really connected to the own experiences of the learner. If we complete the problem by a question which is about the learner, the problem becomes specific and realistic. After that the small child solves the problem about himself/herself, it is easier for he/she to image a situation related to other persons. The problem becomes in this way realistic, natural for the learner.

Marci collects toy cars, Évi collects plush toy figures. Neither of them collected 20 toys. How many toys do they have if Évi has 5 more plush figures than the number of Marci's cars?
How many cars do you have? How many plush toys? Which do you have more and by how many?

Begin the solution of this problem by the collection of data brought from home. Now the children experience how many different number pairs can be given as an answer to the question and perhaps there will be a child in the classroom who has by 5 more plush toys than cars. The number pairs collected in a table format show an example of the purposeful solution of the original problem.

In the next example we have selected the word problem not in order to experience the operational properties, but that the children could see during the problem solution the two types of computation options.

Do you consume 4 liters of milk in one week?

The children begin the solution of the problem by data collection. Every learner can know how many deciliters his/her own home cup in which he/she drinks milk, cacao or other milky liquid is. Here they can discuss how many things are made of milk and children can speak about what others usually eat for breakfast and supper. We can let the children decide about the way of counting. During the discussion it may become clear that from the daily milk consumption we can predict the weekly milk consumption, or we can add to the milk quantity consumed in the morning the quantity consumed in the evening. In this way the unit conversion is made necessary by a real-life problem.

The daily activity of children, their environment and the nature offer a lot of possibilities for the formulation of authentic word problems for small school children. They can collect data about their everyday activities (For example: When do they get up?, When do they go to bed?, Do they have extra classes?, How much sports do they do?...), they can sort the collected data, compare them, formulate questions and can change them. We can also put questions the answering of which requires data completion. The collection of the missing data can be left to the learners, but we can offer options, can make proposals for this.

Data which cannot be completed on the basis of observations, experiences or by measurements require creativity by the learners. The missing data can provoke estimation, or the solution of the problem according to the condition. At the beginning we can accept from the children a formulation like: "in my opinion…". Later they can find several solutions acceptable by them: "May be…, it can also be that…". Ideas collected in groups or frontally can give all possible solutions of the problem.

When doing independent work we can encourage the learners to look for more solutions, or by specifying one or more conditions we can ask them to determine the data specified by the condition.

16 people are sitting altogether at 3 eight-seat dinner tables in the dining room. How many people can have lunch at each table? Look for several possible solutions.

Table 1	8		2	6						
Table 2	6	8			4		0			
Table 3		2	6			7				

Relations, Functions

As in the case of other content areas of mathematics the criteria of a problem's being realistic in the case of relations and functions is also that the learner be able to imagine the content (mostly based on everyday experiences) of the problem.

The basic characteristics of the realistic problems are that they mainly promote the inductive and correlative reasoning in the scope of reasoning skills. Relations observed in the everyday life and working in the fantasy world are created on the basis of finite number of cases, then the produced rule or relation will be valid for the infinite wide circle of the world of phenomena. Compared to the authentic problems the difference is that the problem directs the search for relations and rules and we do not expect that the learner initiate it.

In the realistic problems related to sequences the formal characteristics of the task remain, but the content will be modified that reasoning in the horizontal mathematization starts from the real experiences and from the internal cognitions and the learner tries to find mathematical model to them. In the case of sequences for example the following problems can be regarded realistic by the majority of learners:

Continue the sequence with two members. What can the rule be?

(A) Monday Wednesday Friday Sunday Tuesday ___ ___

(B) January 1 March 3 May 5 July 7 ___ ___

(C) Anna Ágnes Beáta Antal Ábel Barnabás Anita Ágota Bernadett Attila ___ ___

Another field of this topic can be found in the relations between data pairs, that the building up of mental mathematical models is possible by the transformation of the content of the problem with keeping the problem format unchanged. The solution of the following problems requires from the learner to imagine the things contained in them and to construct a mathematical model which can be used in the case of the specific problem. In the case of describing relations between relatives drawing a family tree or any type of tree diagram can make a mathematical model. The visual images of the habitations of animals can be used in the solution by the formulation in words of the analogical relation.

Fill in the chart below.

Father	Younger brother	Great grandpa	Grandpa	
Mother	Younger sister	Great grandma		aunt

Bird	Dog	Man	Squirrel	
Nest	Doghouse	House		stable

The most important general characteristic of the authentic problems is that a kind of problem situation is realized which is connected to the learner's activity and where the learner can act as an active participant. In many cases a kind of "reverse problem setting" can take place, which means that the main point is that in a given problem space the learner has to create the problem himself, or should analyze in what conditions a problem in mathematical sense can be created.

In the case of sequences the basic principle can be that the children recognize patterns, regularities in a given problem space (definition system) and formulate the relations. They should look for examples and counter examples. In this way the authentic problems of relation and functions in addition to the inductive and correlative reasoning are excellent means of development of systematization skill.

In authentic problem situations children with special educational needs should be conducted with more explicit instructions, since without this the contexts and frequent intransparency of the problem make for them focusing on the mathematical characteristics of effects difficult.

In the case of sequences we encourage the learners through authentic problems to search for sequences themselves based on a certain criteria in a well-defined problem space. In the following example the name of the learners define a problem space.

Write on the blackboard the various given names in the class. How can they be sorted? Write down the sorted names.

The solutions can be much diversified. The alphabetical order seems evident, but the length of the name can also be a criteria, or such a refined idea can be used as the sorting of learners' name on the basis of their birth dates. It may happen in every case that the sequence of names will not be strictly monotonous. In this case it is advisable to express the parity relation by the writing of the names one under the other in case of names otherwise sorted in monotonous order.

How could you put the 12 months into order? Find out as many orders as you can.

In the case of the data pairs it is also a possible solution that we draft a two-dimensional data population and the primary task of the learners is to find some criteria based on which the things belong together. It is important to select a basic problem which is natural and relevant for the learners. Such problem areas are for example: school timetable, definitions in connection with meals, relations between relatives, dressing, holidays.

What rule did we use to start to fill in the chart below? Continue according to the rule.

mathematics	Reading	Singing		sports
4	4	2	2	

It is possible that the weekly number of classes is indicated in the table, but may be it is somebody's marks, or how much he/she likes the subjects.

Geometry

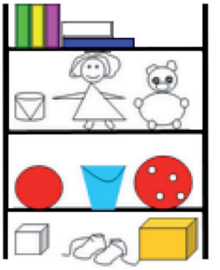
The field of geometry – due to its characteristics – is excellent for the mathematical modelling of things known from everyday life. Geometry is mainly dealing with mathematical characteristics of shapes which can be represented visually, thus it is extremely good for the connection of the visual ideas and mathematical system of definitions. Of the four sub-domains of geometry we first deal with orientation, accentuating by this how many evident possibilities this area offers for the use of realistic texts.

Orientation

It is the purpose of the first grade to lay down the basis of spatial and plane orientation skills using sense perceptions, following directions, changes of directions by movement, comprehension and use of terms referring to determination of places (for example, below, above, next to, between, right, left). In the second grade the formulation of own movements, following routes in reality and on model table, its realization, description of a travelled route, getting to given places, going through given routes in reverse order, impact of change of direction are required. Compared to the expectations of first grade the searching for places characterized by two plane data (direction, distance, vicinity) makes the problems much more difficult.

The shelves shown on the picture are in the Nóri's room. She told us what she had put on the shelves. Write in the missing words.

The shoes are .... the bucket.
The small bucket is .... the two balls.
The bigger box is .... the ball with dots.
The teddy bear is on the .... side of the doll.
The drum is at the .... the hand of the doll.
On the shelf .... the doll there are books.



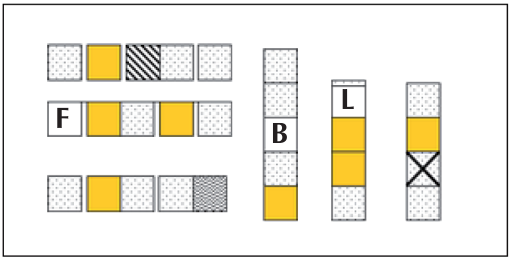
Going-over the route by following the route written by words and by passing by given points.

The drawing is part of a city map. X marks the starting point. Indicate the route.

Look for the house where grandma lives.

- *You leave the house marked by X.*
- *You first go to the Library (L).*
- *After this you go to the Bakery's (B) street along the shorter route.*
- *At the Bakery you buy 5 croissants.*
- *Leaving the Bakery you turn to the right and walk till the end of the street.*
- *You walk around the house whose roof is monochrome.*
- *You turn into the street where there is a house with a wavy pattern on its roof at the corner.*
- *You go to the Flower(F) shop and buy a bouquet of tulips.*
- *Walking around the house of the flower shop you are in the street where grandma lives.*

- *Grandma's house is next to the house with striped roof. But its roof is not monochrome.*



Examples where the task is the comprehension of verbal or written information, the following of directions and changes of directions reflect real situations and cases which are relevant to the learners. The solution can be manipulative or image level. In the case of paper-and-pencil and computer tests obviously the visual problem formulation is an option.

We have hidden a treasure box in the class-room. You will find it, just follow the instructions.

- *Start from the door of the class-room.*
- *Stand opposite to the window.*
- *Take 3 steps ahead.*
- *Turn to the left.*
- *Take 2 steps.*
- *Turn to the right.*
- *Take 2 steps ahead.*
- *The treasure box is at your left leg.*

Work in pairs. Tell your mate the route which leads from your home to the school. Make a draft map. Draw some known places on the map. Your mate should mark the route told by you on the map. Check his/her work.

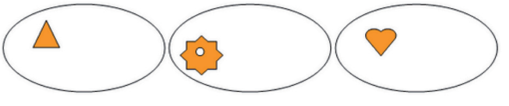
Constructions

In order to develop the abilities required for observing, the comparison of shapes (identification, differentiation, recognition of the shape on the basis of the total view and of one-one accentuated geometrical property) begins in grade 1 and continues in grade 2; also continues the recognition of the part and whole, expression of observations by selection, their formulation by own words, the continuation of the started selection on the basis of the interpretation of properties expressed by words and interpretation of relations. Learners become able to interpret the properties, relations expressed by words. The separation of plane and spatial forms on the basis of their characteristics and their categorization on manipulative and image level, followed by explanations are requirements.

The learners are able to build spatial geometrical objects on the basis of a model. They are able to produce shapes by activities: from mosaic, paper folding, threading of straws, free hand drawing, and later in the second grade by folding right angles, rectangles, squares of paper, copying onto transparent paper, drawing on square sheet, on other nets. Here the creation of forms on the basis of a specified simple condition, as well as the collection, identification, differentiation of works (recognition, naming of some characteristics of polygons: vertexes, number of lateral faces, equality of lateral faces, convexity) are already requirements. All these activities are able to improve creativity, systematization and combinatorial skills. The creation of spatial constructions and plane works with given specificities, and the checking of characteristics contribute to the development of making deductive and inductive inferences.

Example for the sorting, categorization by activities of plane forms on the basis of the observed geometrical characteristics:

Cakes are put on plates in the cake-shop. They began to sort them as shown on the picture.

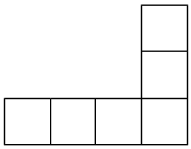


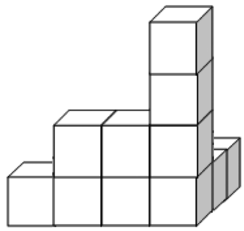
Where will the rest of the cakes be put?
Draw the cakes on the plate where they belong.

Recognition of sensible characteristics of bodies, selection on the basis of identities and differences:

Recognition of bodies on the basis of picture, making of floor plan:

Lali made a house of small white cubes.
Write in the floor plan how he built.
How many small cubes did he use to the house?



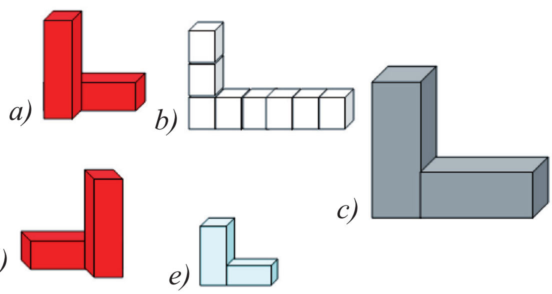


Transformations

The collection of experiences by flat mirrors, the discovery of the symmetry of plane shapes and spatial objects begin already in preschool age, then as a continuation in grades 1-2 the production of mirror shapes and simple mirror image by motion, display, cutting, using copy paper, rotation, or reflection around the axis and by using plane mirror are requirements. Here again the observation (identification, differentiation) comes into the foreground. The monitoring and reformulation of the transformation procedure is important.

It is required to distinguish between the mirror image and the shifted image on the basis of the general view.

Miklós made these houses of identical cubes. Look at the picture. Answer the questions. After the questions write the letter symbol of the house.
Which house is the highest?........
To which house did he use the most building blocks? …….
House A is the mirror image of which house?........
Which two houses are of the same shape?



Distinguishing of the mirror image and the shifted image on the basis of the general view.

Emma has received a new pullover. She liked it very much.

She put it on and went for a walk. She looked at herself in every shop window and in every puddle.

Of the pictures in the second row which one could Emma see in the glass of the shop window?

We suggest using the following example as a task based on practical, playful activity:

Build a house which has a door using the building blocks.

Build the mirror image of each house as well.

You can use a mirror as help.

Measurement

Measurements appear in grades 1-2 connected to the development of number concept. In this connection the main role is given to abilities enabling for comparison and distinction, to the observation, recognition, ordering of correlations: in grade 1 the requirement is the comparison, comparative measurement of different quantities, and the solution of practical problems. After this in grade 2 getting acquainted with standard units (m, dm, cm, kg, dkg; l, dl, hour, minute, day, week, month, year) and the use of their names and symbols belong to the requirements.

Students should observe the relations between quantities, units and measurement index numbers. They use their measurement experiences in estimations and formulate them by their own words.

Father and mother bought a new carpet into the living room. Father and little Gabi walked through the nice, soft carpet hand in hand. Who do you think made more steps, Father or Gabi?

Individual and group project work based on the active, conscious learner's activity give excellent opportunity for the geometrical implementation of authentic problems. In one of the groups of the authentic measurement tasks the learners have to give estimation in situations which are relevant to them. Also measurements made with occasional units, the use of standard units of measurements also belong here - supposing that the problem is not only realistic for the learners, but also relevant.

Estimate how many steps are needed along the length and the width of your classroom.

Choose the shortest child of your class. He/she should measure the width of the room by his/her steps. Measure the length of the room by the steps of your teacher.

What did you find?

Measure the width and the length of the room by a meter rod.

What did you receive? Explain the measurement results.

The following description shows the possibility of a project task:

In your surroundings, search for the symmetrical decorating elements (clothes, furniture, Easter eggs, toys, buildings, trees, flowers, butterflies, churches, pattern decorating eaves, etc.) observe them carefully, analyze the details, record them on drawings, photos, write their histories. Present the results of your research by lecture, exhibition (for example, on posters), by building them (for example, of plasticine (modelling clay), building blocks, gypsum), by making video, etc. The presentation can be made individually or in groups.

Combinatorics, Probability Calculation, Statistics

The foundation of combinatorial reasoning and probability approach is in most cases implemented by games or playful experiments in the school. The child can build into the solution of the problem his/her experiences gained during the game.

For example they play with three red-blue discs[1]. Before every throw they have to guess if the statement will be true or false. The person who makes more correct tips will win.

a) There will be at least two red.

Guess										
Throw										

No. of good guesses: …………

b) There will be at least two blue.

Guess										
Throw										

No. of good guesses: …………

c) There will be at least two of the same colour.

Guess										
Throw										

No. of good guesses: …………

d) Both colours will occur.

Guess										
Throw										

No. of good guesses: …………

1 Such disks have a diameter of about 2 cm, and one of their side is red, and the other side is blue.

In this activity the most important thing for the child is to win the game, therefore he/she tries to use his/her former experiences during the game. From the changing of the tips the teacher can see how the probability approach is improving. For example the fact that there will be at least two disks with the same colour is a sure event. This however will be evident only after making some throws.

During the experimentation we would like to know to what extent the experiences collected by the activities build into the children's thinking.

Therefore a possible version of the above activity formulated by measurement can be the following:

We throw with three discs. Put an X on the right place.

	Sure	Impossible	Probable	Possible
There will be at least wo red ones				
There will be at least two blue ones				
There will be at least two of the same colour				
Both colours will occur				
There will be more red ones than blue ones				
There will be the same number of red ones as blue ones				

In the lower grades during the formation of combinative reasoning and of probability approach we can mention problems which belong not only to this subject. It would be misleading to think that when the primary aim of the school lesson is the improvement of probabilistic reasoning, we only throw dices, rattle coins or pick colour balls out of bags during the whole class. The development of probability approach can be realized in the class-room also by the raising of problems which have relevance to also other fields of mathematics.

Children have to poke on a 0-99 number table blindfolded. Before the starting of the game they have to make a tip if the number can be written as the product of two numbers smaller than 10. (Number 1 is excluded here.)

This game can be played for example when they have to practice the multiplication table. Since they have learned the multiplication tables for a long time earlier – 100 cases separately – there is a great chance to think that there are more numbers in the table which can be found in the little multiplication table than which cannot.

In order to take into account which numbers can be written as the product of two numbers smaller than 10, they cover for example by self-adhesive tapes the fields marked by yellow.

0	1	2	3	2 · 2	5	2 · 3	7	2 · 4	3 · 3
2 · 5	11	6 · 2	13	7 · 2	3 · 5	4 · 4	17	3 · 6	19
10 · 2	3 · 7	22	23	3 · 8	5 · 5	26	3 · 9	4 · 7	29
10 · 3	31	4 · 8	33	34	7 · 5	6 · 6	37	38	39
4 · 10	41	6 · 7	43	44	9 · 5	46	47	6 · 8	7 · 7
5 · 10	51	52	53	9 · 6	55	7 · 8	57	58	59
6 · 10	61	62	9 · 7	8 · 8	65	66	67	68	69
7 · 10	71	9 · 8	73	74	75	76	77	78	79
8 · 10	9 · 9	82	83	84	85	86	87	88	89
9 · 10	91	92	93	94	95	96	97	98	99

It can be surprising how few cases can be found in the little multiplication table, therefore there is not much chance that we poke the desired number (we poke numbers which can be written as a product where both numbers are bigger than 1 in only 36 cases of 100).

In this game children can gain experiences about the commutativity of multiplication and can look for numbers which can be written down in many different ways as a product. The observation that what happen more often in more different ways is more probable can change their probability approach.

They repeat the same activity later, but this time they are looking for numbers which can be written as a product. (For example 33=11 · 3) In this way there is a much greater chance that we poke a number which can be written as a product. A game like this offers the first opportunity for the children to gain experiences about prime numbers. It is not the teacher who mentions the problem, but there is a strong urge in the child to follow up the problem. The systematic search for complex numbers can be the basis of the procedures aiming at the screening of prime numbers (for example Sieve of Eratosthenes).

In the school the children poked blindfolded at a table with numbers ordered from 0 to 99. The winner was who poked a number which could be written as the product of two numbers smaller than 10, but bigger than 1. Colour in the winning fields.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

What do you think is the more likely outcome of a game? Underline the correct answer.

Winning is more likely *Loosing is more likely*

Reasons for my answer: ....
....
....

Based on the reasons we can have a view about the development level of the child's probability approach. Based on the answer it becomes clear whether the child feels the fact that what can happen in different ways is more probable.

Detailed Assessment Frameworks of Grades 3-4

Numbers, Operations, Algebra

The practical use of mathematical knowledge about numbers, operations and in algebra is an extremely important field of the mathematical reasoning. It is also the task of teaching mathematics to show the indispensable role of the subject in other disciplines and in the everyday life. The examples taken from other subjects and from the practical life prove the usefulness of mathematics for the children. We can provoke the children's interest and curiosity towards mathematics by diversified problem settings. Therefore the selection of the subjects of problems requires discretion. It is not only the mathematical model which determines the difficulty of the problem solution. The same mathematical problem can be difficult to varying degrees for the children if we present them in different context. Therefore in the analysis of the problem solution we also take into account what is difficult for the learner. The selected model, the drawing made by the learner can give information about the comprehension, about the recognition or misunderstanding of the relation formulated by the text. By making a proposal or giving an instruction for the use of a certain model we can promote or just the opposite we can make difficult the solution of the problems. In this case we want to check not only the understanding of the problem, but also the problem solution with the selected model. We can expect from the learner the successful problem solution with the use of an optional or given model if we paid enough attention to this with the diversified solution of problems, with their comparison, with the discussion of the advantages and disadvantages of the selected solution method.

For example:
In the flower shop one daffodil costs 60 Ft, one tulip costs 80 Ft. We bought the same quantity of the two sorts and paid 420 Ft. How many flowers did we buy of the two sorts?

Visual representations promote understanding, the revealing of relations and correlations which are indispensable parts of problem solution. Therefore the improvement of the children's model making ability is very important. Different models can help the recognition of the contexts, the tools can be for example, demonstrating by means of manipulating objects, drawings, open sentences, tables, representation by segments, number line.

The solution of the first task by using tokens is evident. For example children draw a daffodil and a tulip and put the corresponding sums on the drawings. They do this until they get to 420 Ft.

Children having better abstraction skill can solve the problem with the help of a table. For example they can make a table like this:

Number of tulips and daffodils	1-1	2-2	3-3
Price of daffodils	60 Ft	120 Ft	180 Ft
Price of tulips	80 Ft	160 Ft	240 Ft
Amount to be paid	140 Ft	280 Ft	420 Ft

In this solution from two known data we arrived at the amount specified by the problem with systematic trying, evenly increasing the amount to be paid. In the meantime we calculated data which are not necessary to answering the original problem. In the previous table the evenly increasing sequences can be recognized, the new data obtained give new information.

For example:

- The value in row 2 and column 6 of the above table is an answer for what?
- What can we learn from the data in column 8 of the last row?
- What does the sum of numbers is row 2 column 3 and in row 3 column 2 mean?

  …

The children can also choose open sentences to the solution of the original problem.

They can say: 1 daffodil and 1 tulip costs together $60 + 80 = 140$ Forint. We do not know how many pieces we will buy, therefore we shall mark this by: $\square$

We pay as many times 140 Ft as many tulips and daffodils we ask and this costs 420 Forint. We can describe this with the following operation: $140 \cdot \square = 420$

The solution of the open sentence can be looked for by estimation, by trying estimation then by its correction, for example in the following steps:

The value of 140 rounded to hundreds is 100, and of 420 it is 400. We have to take 100 4 times so that we get 400. Testing shows that $140 \cdot 4 > 420$, therefore we have to try with a number smaller than 4. Trying number 3 we find that the equality is true $140 \cdot 3 = 420$.

In the solution we intentionally wanted to answer the question. We did not get other information, we cannot formulate new questions which can be answered easily. Every new question requires new open sentence and its solution.

In a housing estate there are identical ten-story houses. In every house, i.e. high-rise apartment block, there are 6 apartments to the left and 8 apartments to the right of the staircase on every floor above the ground. On the ground floor, there are shops. In this housing estate there are 420 apartments in total. How many houses are there in the housing estate?

The problem can be well solved by making a drawing, by creating a tasks consisting of numerals only, or by an open sentence. Certainly, we expect simplified drawing from the children by indicated the most necessary data. For example:

6 apartments		8 apartments

They can think in different ways. For example: In this house there are totally 140 apartments of the 420, on the left side of the staircase 60, and on the right side 80. The rest of the apartments (420 – 140 = 280) are in the other houses. There are also 140 apartments in the second house, the remaining apartments can be found in the third house: 280 – 140 = 140.

In this solution we moved from the known data towards the solution. In the different steps we got answer to the questions how many apartments were in the housing estate, if by 1 or 2 fewer houses were built.

Also starting from the total number of apartments children arrive at the solution through the following steps: if every house has 10 stories, and there are the same number of apartments on every level, on one level there is one tenth of that amount, that is: 420/10=42 apartments. In each house there is 6+8=14 apartments on a level, therefore the number of houses is equal to the division of 42 by 14. The result of operation 42:14=3 means the number of houses. In this case we arrived at the solution through two number problems and we obtained one plus information, namely that there are 42 apartments per levels in the housing estate.

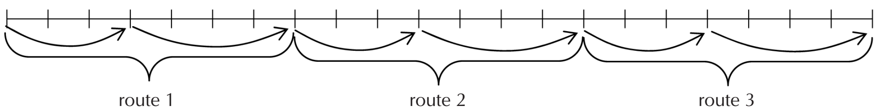
Children can use the open sentence method to the solution of this problem, too: in a house there are 6+8 apartments on a level, on the ten levels there are 10 times more, that is (6+8)·10=140 apartments. Mark the number of houses by □. □ houses multiplied by □ times 140, that is there are 140·□ = 420 apartments. The finding of the solution can be made in the same way as described in the previous problem.

A bus driver travels between two cities. He covers the distance from city X to city Y in 60 minutes, and gets from Y to X in 80 minutes. How many turns did the bus driver take on the day when he drove 7 hours?

The solution of the previous problem is followed by the discussion why the travel time is longer into one direction than into the other. Children can find the answer for the question based on their experiences. For example:

- Bus goes on a longer route from Y to X.
- There are a lot of slopes when bus goes from Y to X.
- In one direction the bus works as express service, in the other it stops on many places.
- In one direction it goes on motorway in the other on highway.

The story can be illustrated by a time line where the 7 hours are divided into for example, 20 minutes intervals. Children can indicate the time passed on the line. For example:



This figure can also give new information. Children can put and answer questions themselves. For example we can get questions like:

- Where was the bus driver after 200 minutes driving?
- Where was the bus driver when he said: “I have already driven 3 hours.”
- How many times had he already driven when he departed from Y city to X city?
- During the day when had the bus driver the chance to have a rest?

…

The above figure shows well the mathematical models which can be the aids to the solution of the problem with real content. A sequence of changing difference can be red about the arrows: 60, 140, 200, 280, 340, 420…

Braces include two arrows and illustrate the mathematical content of problems like one arrow instead of two. Based on this we can read the members of an evenly increasing sequence: 140, 280, 420…

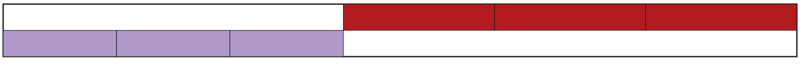
These numbers get meaning by the fact that we relate them to the problem and we say which number informs us about what.

The time line shows continuity well, with the help of it we can have an approximate view about the staying place of the bus driver.

Cities A and B are 420 kms apart from each other. Two cars depart from each city towards the other one at the same time. The car starting from city A makes 60 km per hour, the one starting from B makes 80 km per hour. When and where shall they meet?

The problem can be well solved by a 42 cm paper strip and by the lilac and claret rods of the Cuisenaire-type[2] rod set.

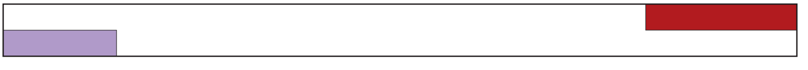
Children can represent the routes travelled per hours with the help of these tools.



What information can be read from the display?

Children follow the way of cars in head.

They imagine what route did one car made during 1 hour and where did it get and see well what distance is left between the two cars. They can find intelligent explanation for both number problems of 420 – 60 – 80 or 420 – (60 + 80).



They can also easily read from the picture how long distance is left for each car from the total route. They can even find answer for question like where were the cars half an hour before.

2 The colors assigned to the rods are different in Hungary from the original Cuisenaire standard. Here lilac refers to the 6, claret refers to 8.

If they travel the whole route by the cars they can again get a lot of information.

On the one hand they can see how the cars get farther from each other after the meeting. It can be followed well that the car leaving city A makes the whole route in 7 hours, while the car starting from B needs only somewhat more than 5 hours. The smarter children can even calculate during exactly how many hours the car gets from B to A.

The last two problems contain quantitative data, but their solution is more difficult for the children. Therefore in grade 3 we spend more time on the discussion of word problems about motions, as a model of which we can use illustration by segments, in addition to colour rods and paper strips.

It is clear from the above-presented solutions that in addition to the solutions other information can also be read from the displays and illustrations the great advantage of which is that they strengthen the detection of the relations between mathematics and real life.

For learners of grades 3-4 we often set problems which they can often meet in their everyday real life situations. To the solution of these problems they need to apply the acquired mathematical knowledge, to mobilize their various skills. Children have the feeling that the everyday problems telling real stories are close to them, since they can have the impression that the moments of their own life, their activities are livened. These stories make possible that the children live through these situations with empathy and they obtain knowledge which can be used and easily activated in the everyday life. The methods of differentiated problem settings offer possibilities for active learning, for the discovery of relations and make the learners to activate their reasoning.

The real problems taken from the children's life and environment help them to recognize the model role of mathematics in the solution of the problems raised by real life or by different disciplines. Activities requiring the measurement of quantities and problems about shopping contribute to this.

Children can also often get the task to go for shopping. Many problems can belong to this activity. The problems may be connected to the payment of the purchased goods, to the delivery of goods and the estimation of the weight of several goods

During a big shopping trip we have put a lot of things into the shopping trolley.

Items:		*and their prices:*
1 carton of milk in paper box	1liter	93 Ft
One and half kilogram of meat	1 kg	768 Ft
4 boxes of eggs	1 box	12 Ft
400 grams of cheese	1 kg	720 Ft
2 boxes of 250 gram – chestnut purée	1 box	174 Ft
3 dl of cream		105 Ft
3 kg of washing powder (detergent)		1300 Ft
4 kg of apples	1 kg	150 Ft
2 kg of mandarins	1 kg	280 Ft

a) Going to the cash-desk, we are pondering whether the 8000 Ft we have in cash will be enough or we should pay by card. What do you think?

b) We have put everything into two bags except for the milk, the washing powder, the apples and the eggs. How could we distribute the items into two bags that if their weight is nearly the same? What would you put into the first and what into the second bag?

The solution of the problem improves various abilities. First, there is a need for the estimation about real-life data and also for the measurement of quantities (for example, how heavy one box of eggs is?). Some data are missing or are unknown to the children, these data have to be added. For example: How many liters of milk are there in one carton? How many eggs are there in a box? When adding the data the children see that the solution of the problem is not clear, since we can get eggs in many different packages. Thus the amount to be paid depends on how many eggs we bought. This however does not influence the weight of the bags, since we do not put eggs to the bag.

In the everyday life we often get into decision-making situations. In general there are different options for the solution of a problem and it depends on our choice how we solve the problem. Our choice can be influenced by a lot of factors, the solution depends on different conditions. Therefore we need to bring the children in situations on the mathematical lessons where they have to think over the possible conditions and in case of meeting several conditions, they will select the most realistic solution.

Hopefully reading is part of the children's everyday life. In addition to the discussion of the reading experiences it is also a possibility that they find ideas for the solution of the technical problems. This can be the sorting of books, their placing on a given shelf, borrowing from the library, or the scheduling in time of the reading of a book.

For example:

Andris likes to read very much. He reads every evening one hour before sleeping. One of his favourite books is Kele from István Fekete. He borrowed it for the third time from the library but has to give it back in one week.

a) He already read half through the 270-page book but had not arrived at its two third yet. At least how many pages of the book does he have to read every day so that he could finish the book in one week?

b) Andris made a note about the opening hours of the library.

Monday	*10:00–12:00 and 15:00–16:30*
Tuesday:	*14:30–18:30*
Wednesday:	*11:00–17:15*
Thursday:	*9:30–11:30 and 15:15–18:00*
Friday	*10:00–13:30*

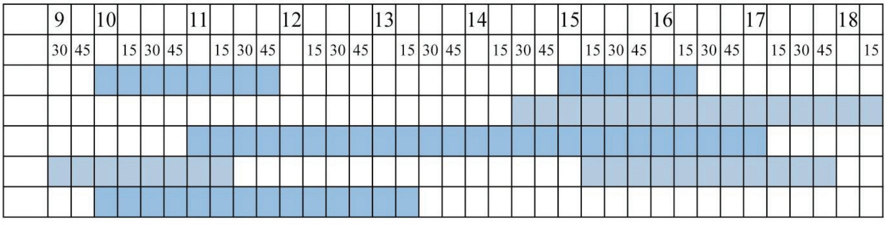
Because of his school schedule and sports program Andris can go to the library early afternoon, between half past one and 2, or after 5 o'clock in the evening. On which days can he return the books to the library?

The first part of the problem tells us that we have less than 135 pages left, but we still have more than 90 pages to read in the book. If somebody wants to read this during a given period of time more specific data is needed. Thus we can only think over how much one can read during one day if 134, 133, …, 91 pages are left from the book. We can also consider that the problem still does not have 44 solutions since Andris every day reads the same quantity, thus if the number of pages read every day increases by 1 page, the number of pages red will increase by 7 pages during 1 week. Thus it is worth to collect the possible solutions of the problem in a table:

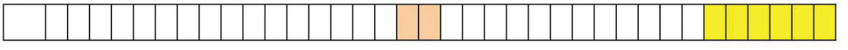
Number of unread pages	91	92–98	99–105	106–112	113–119	120–129	130–134
One should read this much during 1 day	13	14	15	16	17	18	19

One should also consider in case of how many pages is it necessary for Andris to read the same quantity of pages on every day of the week and in which case are less pages left for the last day.

The second question mentioned in the problem requires the comparison of time intervals and the determination of their common part. The representation of the opening hours of the library on a time strip gives great help to the children.



We can also represent on a time strip how many times Andris can go to the library and we can cut this strip out and pull it along the table.



By moving the strip the children can easily read the possible solutions.

The correct choice of different tools makes the activity of "mathematization" easier for the children. It is an important and difficult task to develop the children's ability to translate the problems formulated in everyday language to the language of mathematics. The translation is supported by tools and carefully selected pictures. The tools proposed for the activity are similar to the real-life objects (for example, measurement, invoicing of specific objects, use of tokens), in other cases pictures, figures (for example, representation by segments), or models leading to abstraction appear (for example, colour rods, table). We should not urge making solutions without using tools, it is important that we often use the reasoning by illustration. Before using algorithm we should expect from the children for example, the estimation of operations, in this way they can check the reliability of their calculation, they can recognize the mistakes made.

Problems containing natural, geographical, climate data or connected to excursions, travels, sports, healthy way of life prove that mathematics is a useful means in solving problems of other disciplines and of other fields of life.

The problems should set realistic situations which the children meet day by day, thus it will be easy for them to imagine the situation. When selecting a topic it is not the mathematical problem to which we are looking for a nearly realistic situation, rather we formulate real problems which often happen in everyday life and by thinking about these problems we help the children to get easier orientation in the everyday life.

We can pose problems where we expect from the children the collection of the required data.

For example:

Collect data about yourself.

a) How many times does your heart beat in 1 minute?

b) How many times do you take a breath in 1 minute?

Calculate.

c) How many times does your heart beat in 1 hour?

d) How many times do you take a breath in 1 hour?

The children have a simple problem in front of them which they can solve in one step. There can be big differences between different solutions of a problem, since the collected data can change based on the children's measurement results. The comparison of the solution can screen the incorrect measurement results in this way it leads to the use of realistic data.

The solution of the problems independently, in pairs or in teams allows the review of problems different from the customary ones and the consideration of real life situations, the recognition of solution methods, collection of ideas, methods, and it contributes to the improvement of creativity which essential to the problem solution. During the team activities the children learn the rules of co-existence in a natural way, they experience the good feeling of helping somebody. They have a lot of opportunities for expressing their opinions, to let others know their ideas. They are trained to respect the opinions of others and to being tolerant to their fellows through the confrontation, discussion of views. They learn how to accept the imperfections, eventual limits of their own and of others. The correction of mistakes, giving opinions and persuading others about their goodness of the own ideas can be implemented by reasoning, using rational, acceptable arguments. We have to ask the children to check the solution and to explain why they selected the given method so that the children take responsibility for their work supported by facts and be able to evaluate their activity in a realistic way.

Relations, Functions

Based on the requirements of grade 1-2 similar problems and requirements can be set by the end of grades 3-4. The recognition of more complex rules in the case of sequences is a requirement and we can suppose greater skill in the re-coding of mathematical features of everyday objects and phenomena. For example the transformation into numbers of facts in connection with time can become a routine, because for example the names of the days of the week and the fact about which day of the week we are talking about is already present as a factual knowledge element at this age and it is not necessary to use a strategy similar to the practice of counting starting from Monday.

The children can meet recursive sequences of numbers already in grades 1-2, too (for example, with sequences where the next member is the sum of the previous two members), there are however many possibilities for the formulation of the everyday problems where recursive sequences appear. For example in how many ways can a 2/4 musical cadence be filled by quarter and eighth notes? Then: in how many ways can a ¾ musical cadence be filled by quarter and eighth notes?

The next problem is the text version of the classical Fibonacci sequence, using a drawable wording, recalling the world of tales instead of the unnatural growth of rabbit population:

When the oldest tree of Fairyland was planted, the tree had only one branch. One year later the tree still had only one branch, but then in each consecutive year a new branch grew. How many branches had the tree in (a) two years, (b) three years, (c) four years, (d) eight years?

What can be the rule in the following table? Mark the symbol of connection that is true for the table and cross out the one that is not true.

△	Wheat	House		treasure
☐	b	h	f	

a) ☐ = of the letters of △ we leave out the ones that are not initials
b) ☐ consonant
c) ☐ = initial of △
d) ☐ letter

In this example all the four options are true for the table.

These types of examples – although they are less customary – measure the high level components of cognition which are connected to the falsification principle.

What can be the rule in the table below? [This exercise is specific to the Hungarian language.]

☉	lent	mellett		mögött
□	lefelé	mellé	alá	

(The first row concerns directions and the second row concerns positions of the according directions.)

From content point of view the problem is grammatical, but it illustrates the mathematical laws of the grammar. This strengthens the relations between mathematical modelling and the knowledge obtained in the everyday life and this is a specific objective of mathematical education which has the aim to develop mathematical cognition and to make the mathematical knowledge transferable.

What can be the rule in the table below?(Source: Az általános iskolai nevelés és oktatás terve, /Concept of elementary school education and training/ 1981, Edition 2, p. 278)

□	Horse	Bear	Cow	Hen
✷	Colt	Cub	Calf	Chicken

Formulate the solution by an open sentence, too: the young of □ is ✷.

There are a lot of means for understanding binary relations. In a lot of school subjects the closed matching problems are regular problem types, when relations have to be found between the elements of two sets and it may happen that several elements of the set can be matched with the elements of the other set. Questions in connection with the agenda, eating, and clothing make possible the making of data pairs, where the order and the relations between the prefix and suffix are also important in their connection.

We can define authentic problems by the use of requirements and problem types of grades 1-2, but by moving in a wider number circle. In addition to data pairs correlations recognized in data triads can also be expected.

Problems requiring selection according to two aspects can be found among tasks for developing the systematization ability. The systematization of a given number of things by the projection of two aspects on each other can be solved already in grades 3-4 mainly by manipulative and pictorial tasks, the content of which is well-known to the learners from the everyday life. The two-way classification is at the same time a means of improvement of correlative reasoning, since in the two-dimensional system produced by the projection on each other of the two aspects the eventual correlation between the two aspects becomes evident.

In the case of similar problems the cooperation of the learners in heterogeneous groups formed according to students' ability levels can be proposed from instructional methodological point of view so that they could learn the ideas of each other.

This is especially important in the case of authentic problems that do not have only one, well defined solution, but in many cases the solution itself is the reasoning process in the course of which mathematical models develop and change.

The principle of inverse proportionality also appears in grades 3-4 mainly in problems based on the experiences, trials of the learners.

During the class excursion the children wanted to travel in a little carriage pulled by a pony. The owner of the pony said that they have to pay 1200 Ft for a quarter of an hour ride, irrespective of the number of passengers.
What other questions did the children put before they rented the coach?
Write the price per chind of the ride into the following table if they travel single, or in two, three, or four of them together.

Number of participants	1 pupil	2 pupils	3 pupils	4 pupils
Fee/participant	1200			

The other possible occurrence of inverse proportionality is the computation of area. (Certainly we do not think of area computation described by a formula.)

Anna would like to arrange 24 identical paper boxes in her room. If she puts them one on top of the other the column will be high, if she puts them one next to the other they will occupy a lot of space on the carpet. What kind of arrangement would you suggest? How many boxes should we put next to each other and how high should Anna put the boxes? Make a drawing then a chart.

No. of boxes next to each other	1	24	2		
No. of boxes on top of each other	24	1			

The problems where two features which can be determined by numbers are correlated not in deterministic way, but a tendency is outlined are good for the development of correlative reasoning. In the following example the learners can also use their own data.

The health assistants measured the height and weight of the pupils of the class. Some data can be found in the following table. Two pieces of data were however erased by somebody by mistake. What data can be put into the empty places?

Height (cm)	135	142	127		140
weight (kg)	31	36	28	40	

The actual data can be taken from a relatively wide interval, but the explicit formulation of the correlation between the data rows is much more important, which can be the description of positive correlation in the language of children. Even more important is however the cognition, that the specific value cannot be determined on the basis of correlation.

Geometry

Constructions

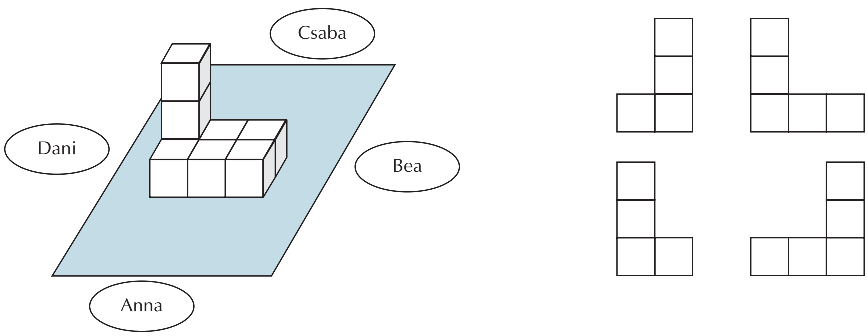
In grade 3 creative thinking comes into the foreground together with the development of vision of forms and spaces. During the creation of geometrical objects text comprehension (by verbal expression of geometrical features), observation skill and students' memory develop. Abstraction is improved by observation, and by description of the properties of specific forms.

By the end of grade 4 the construction of objects is enriched by taking account of specified conditions and by checking them. The understanding of the relations between parts and whole, the analysis, formulation of observations, and the basic use of the acquired mathematical language becomes evident. Combinatorial reasoning is developed by the creation of works. The aim is to reach completeness and to build up the system of creations.

For the upper grades the building of objects after models or the production of plane shapes by activity according to given conditions are necessary prerequisite knowledge items. Further requirements: The recognition of geometrical properties, the selection of shapes, their sorting on the basis of recognized features. Recognition and taking into account of edges, vertexes, faces in case of simple bodies, recognition and taking into account of sides, vertexes in case of simple polygons. Recognition of cuboids, cube, and rectangle, square based on total view in different positions of bodies and plane forms. Listing the learned properties of rectangle, square, cuboids, and cube – with the help of a model shape or drawing.

Task for the improvement of spatial vision, of the observation of relations between part and whole:

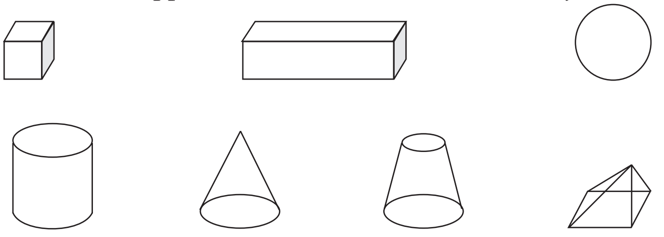
Four children have built a solid of four small cubes. They sat around it and made drawings from four different sides about what they saw. What do you think the children could see. Write their names on the correct places.



Example for the recognition of forms, for the recognition of the properties of shapes and for the determination of the trueness of statements:

Play in pairs.

Select solids you can see in the picture. Pick a card from a deck of ten *cards, taking turns. Supplement the statement on the card so that you pick up a solid for which it is true. (You can play the game according to different rules, too: Supplement the statements so that they become false.)*



Make cards with the following sentences. Only one sentence should be written on one card.

All of its faces are of the same form and size.
All of its faces are square.
All of its faces are rectangle.
It has 12 edges.
The number of its vertices is 5.
The side faces are quadrilaterals.
It also has round shape face.
It only has curved surface.
It has both plane and curved surfaces.
It has triangle shape face, too.

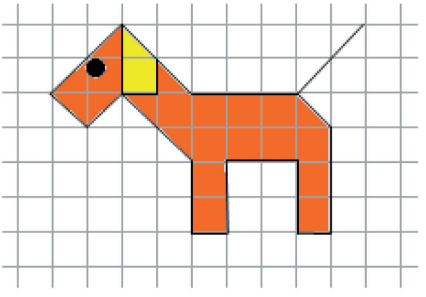
Transformations

In grades 3-4 in addition to the identification of different geometrical objects, the recognition and making aware of changes and invariability lay the basis of generalization in the field of transformations. The recognition of rhythm, periodicity, the observation and following of symmetries aim at the improvement of observation skills. It is important to formulate statements about the observed shapes, or to determine the trueness of given statements.

The development conditions needed to the progress require the recognition of "similar" and "congruent" relations, laying the basis of visual definition of similarity and congruency, implementation of two-dimensional congruent transformations (translation, reflection around the axis, rotation) with the help of copying paper, differentiation of mirror image and translated image in the case of more complex forms.

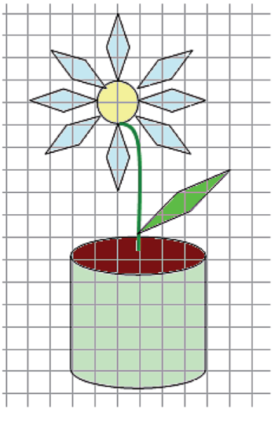
An example of similarity when producing an enlarged image:

Grandma is embroidering a dog on Danika's blanket. She has found a pattern in the "Skilled hands" journal, but its size is too small to the blanket. Copy the pattern into your exercise-book. Enlarge it by doubling the units of lendth in both directions.



In addition to the recognition of the "similar" and "congruent" relations an important element of this topic is the creation of mirror image around the axis with the help of copying paper, or the production of enlarged picture using a quadratic grid as examples of plane congruent transformations.

You can make a nice gift for Mother's Day.
Take a strong paper box.
Enlarge the drawing to its double,
and glue it on the top of the box.
It will be a good sewing box.



Orientation

In grades 3-4, the further development of spatial orientation, the understanding of information, elaboration of geometrical information by words or symbols, the development of ability to determine places are all considered means for ability development. Similarly, remembering directions, dimensions, vicinity.

The further development requires that the learners be able to orientate in their own environment (street, house number, floor, door, direction, distance), interpret or prepare simple draft maps with the approximate indication of direction and dimension, and the precise definition of vicinity. They should be able to orientate on line, in plane, in space with the help of one, two or three data.

Below please find the following example of a task built on authentic activities:

> *Work in pairs. Make a draft map about the environment of your school. Mark the places of the school, the shops and of your home. Indicate on the map if there are parking places, railway station, sports ground, library, cinema, theatre around. Give a route description what your pair has to follow. He/she has to tell where he/she has got to. Then he/she will give you a route plan to follow. Where did you get?*

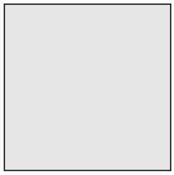
Measurement

In grades 3-4 the collection of experiences in the field of recognition, differentiation of quantitative characteristics, detection of differences is continued. The task is to improve skills necessary for giving estimations, to express the extent of precision in practical measurements, to make simple quantitative deductions. It is important to build relations between mathematics and real life. The practical measurements contribute to orientation in quantitative characteristics of the world.

In order to lay the basis of the developments in the consecutive years the learners should be able to make measurements and use the occasional and standard measurement units. Based on practical measurements they are able to understand the relation between unit and index numbers, to make conversions with the learned units of measurement in connection with practical measurements and to determine the perimeter and area of a rectangle (square) by measurement and computation.

Example for the construction of a rectangle, and determination of its area:

Uncle János has paved the sidewalk with square shaped flagstones. With the remaining 36 stones, he wants to pave the ractangle shaped place in front of the kennel. He is trying out how to place them on a piece of paper. Make a drawing of all the possible solutions.
In reality the length of one side of the square-shaped flagstones is 1 dm:



How many square decimeters of area could he cover with the remaining blocks?

Additional ideas of examples which are activity-focused and are directly related to the everyday experiences of the learners:
The wind has slammed the window and unfortunately, has broken the glass. The housekeeper measured that a 1253 mm long and 1245 mm high glass sheet should be cut into the frame.
a) Measure and cut out a paper sheet of this size.
(You can glue together several pieces to create this sheet.)
b) Give the dimensions with centimeter precision.
c) How many centimeters is the total length of framing strips around the glass sheet?
d) By how many pieces of 1 cm side length square could you cover the glass sheet?

Combinatorics, Probability Calculation, Statistics

In grades 3-4 the main emphasis is placed on the more conscious way of gathering and interpretation of data. By the end of grade 4 children are able to arrange data in a sequence or table, they can represent them of diagrams, they can read data about diagrams, sequences, tables, graphs and they can find data representing a whole data set (for example, the middle one according to size; the biggest, smallest data and their distance; the most frequent data). They can calculate the mean value of the data. This topic gives a lot of opportunities for the solution of realistic tasks if we carefully select the data.

We often examine the setting and solution of realistic problems with the help of the ability to analyse tables. For example:

Gabi, Béla, Pista and Jutka are very good friends. They like to play cards, therefore they play at least once per month. They play a game where there is always a first, second, third and fourth place. They write down the results after every round and at the end they announce the winner at the end of the year. The following table shows the results of this year's tournament.

	1st place	2nd place	3rd place	4th place
Gabi	12	24	23	17
Jutka	18	22	21	15
Béla	24	13	13	26
Pista	22	17	19	18

During the interpretation of the table the following questions emerge: How many games were played this year? How can this be counted? About how many rounds are played on each occasion? Who won the most games?, etc.

Another problem can be that if we select the winner on the basis of other aspects, another competitor will be the winner. Children are able to look for rational criteria based on which it can be determined who can be regarded the winner.

The explanation of the students, the dispute gives a hint that the statistical data set can be interpreted and explained in several ways, since

- in Béla's view he is the winner, since he has won most of the games.
- Jutka has the opinion that although she has not won so much, but had very few last places.
- Gabi is of the opinion that she herself has won very few games but was on the second place many times what is very difficult to make. And she has less last places than the boys.
- Pista feels that he is at least better than Béla, because although he has less first places, but less last places, too.

The solution of the problem can be for example that they give 4 scores for every winning, three scores for the second place, and two scores for the third and one for the fourth places. It is possible that they think that more scores can be given for the winner. For example 5 scores for the win, three for the second place, one for the third and nothing for the fourth place. Would both ways of calculation produce the same result, the same winner? In addition to the orientation in the table an important advantage of the activity is the improvement of computation skill.

A little more simplified example of the above problem can be:

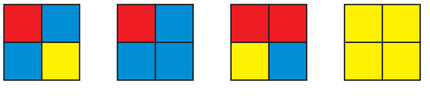
The school football championship is over. The teams got 2 points for a win and 1 for a tie. The results of the matches were written in the following table:

	3.a	**3.b**	**3.c**	**4.a**	**4.b**
3.a		3:0	2:1	1:3	1:1
3.b			0:0	0:2	2:1
3.c				4:3	1:3
4.a					2:2
4.b					

Determine the number of points each team collected.
3.a:……points
3.b:……points
4.a:……points
On which match did they score the most goals?
How many matches ended in a tie?

Children meet a lot of authentic problems during the learning of combinatorics and probability calculation. Built on their everyday experiences a lot of problems can be set which are practical, relevant to them and are intransparent as to the problem solution process.

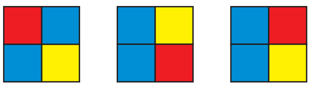
A good occasion for this is for example the creation of a set of toys by a team work.



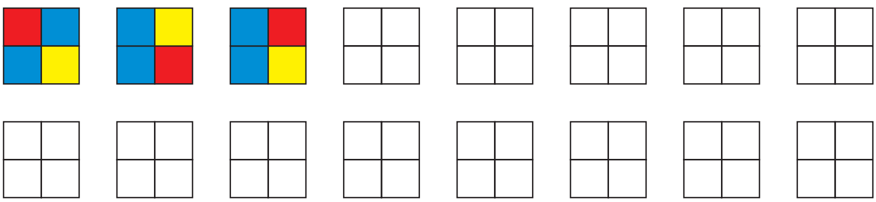
We draw the middle lines of squares and paint them using red, yellow and blue colours. The shared task of the children is to create all possible different elements. Since the papers can be rotated we can agree that we regard the sheets which can be translated into each other by rotating around the centre point of the square. In this case both the organization and division of the labour pose a combinatorics problem. Children regard the prepared set as their own and this makes the activity authentic.

The version formulated in the course of assessing the above-written activity can be the following:

We make the following puzzle using painted squares. We used three colours.

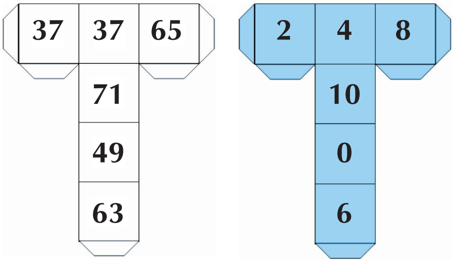


Next, we separated the group where all three colours were found. What other elements can be found in this group? Colour.



The development targets of the probability approach contain that gathering, observing and processing of data in grades 5-6 should be made more and more without the teacher's help. This contributes to the development of systematization ability and makes possible the observation of frequencies, too. The statistical observations offer a lot of opportunities for the insertion of authentic tasks.

In another case they throw simultaneously with the two dices shown on the picture below. Before the starting of the test they formulate some guesses (for example, whether the even or odd sum will be appearing more often) and after the putting down of some cases they compare their experiences and the guesses.



The questions separating the sure and impossible events are still very important during the measurement. We can ask the following when throwing up the above dices:

I threw these numbered dice, then I claimed things about the product of the numbers thrown. Write next to the statement if in your opinion it is true (T), false (F) or can be true, but it is not sure (C).
a) ended in 4 .....
b) smaller than 6 ....
c) odd....
d) 491...
e) smaller than 711
f) ....

There are good opportunities for describing authentic problems when children have to plan the rules of a game themselves.
For example:
Jancsi and Peter take turns throwing a regular dice five times. They agree that Jancsi scores one point if the result is 2, 3, 4, 5, or 6. Peter gets some points otherwise. After throwing five times the winner is the one who could collect more points. How many points should Peter get when the dice shows 1, if we want the game to be fair?

As an expectable solution children will propose 5 or 6 scores to be given for throwing 1. The teacher does not have to take a position about the final, correct solution. The problem follows the idea of the so-called problem-based learning that is the learners make mathematical activity on an intransparent problem, while the teacher becomes the facilitator and moderator of students' reasoning stepping down from the role of the owner and distributor of mathematical truths.

Detailed Assessment Frameworks of Grades 5-6

Numbers, Operations, Algebra

Compared to the former grades the word problems can bring a lot of novelties in the assessment of students' knowledge. In the extended number circle quantities not connected to practical experiences but known from the media or from the school material (for example, historical years, geographical quantities) can be included in the problems. In addition to this the problems to be solved in several steps also gain greater space. The majority of steps consist not necessarily of more arithmetical operations to be performed one after the other (although this also creates a lot of difficulties), but of the sequence of the conscious decisions appearing in the different phases of the problem solution process. Certain steps become especially important in the realistic problems. The understanding of the text of the problem and the selection of the correct mathematical model are in general of greater importance than in the test problems. Also of outstanding importance is the interpretation in general, control step of the problem solution, which does not mean here that we perform the completed mathematical operations again, or compute them with their inverse, but that we test the matching with the problem's text and the conformity with real life.

In our introductory chapter about the application of mathematical knowledge we presented several examples of the realistic arithmetical word problems. Of these prototype problems other realistic word problems can be generated.

In the apple garden of Uncle Jancsi the fruit trees are in 8 rows and 12 apple trees can be found in each row. At his son's suggestion he treats the trunk of the trees at the edge of the garden by chemicals to keep the roving deer away from the trees. How many fruit trees will not be treated by chemicals?

It is proposed to prepare a draft drawing to the solution of the problem that is we connect the things in the problem's wording to a geometrical model.

It is printed on a cinema ticket that "LEFT, row 17, seat 15". How many seats can be found in the cinema?

From the point of view of intransparency this open-ended problem can even be placed among the authentic problems. Several different estimations can be given as a solution, which can be formulated as inequalities with mathematical symbols.

280 pupils are transported to the Children's Day celebration in 44-seat buses. How many buses should be ordered by the headmaster of the school?

International experiences were collected about the problems where some kind of "trick" is hidden. Probably the majority of children can compute correctly the division with remainder the result of which is 6 and the remainder is 16. Many children will however give an answer of 6 or they may also give the answer „6, 16 is left ". The realistic answer here will be 7, to which we use the implicit information that obviously they will order the least possible buses.

Because of the data not contained in the text of the problem or due to the factors typically not regarded mathematical the learners often feel themselves cheated when they solve problems, like for example:

The best result of Jancsi in 100 meter run is 17 seconds. How long would it take for him to run 1 km?

Our proposal is that these types of "tricky" problems have a place in the school lessons, especially in order to avoid the over automatization of the usual problem solving strategies, they can however hardly be used for diagnostic assessment purposes because we can only reveal using other fine tests if somebody answers 170 seconds to the above problem because of being uninformed, or because of a lack of courage.

It is an important step to the better understanding of the word problems if we often expect from the learners of this age to find out word problems by themselves for a given mathematical structure. This is an extremely difficult task. It may be a difficult work to create a text even to one basic arithmetic operation. But if the children are allowed to create texts to the problems this allows the putting next to each other and the comparison of the routine word problems and the realistic problems.

If for example 20 liter water should be divided equally in 8 vessels, formulating this as a routine word problem the end result of 2,5 liter comes easily. We can ask the learner what other elements can be put into the problem so that the numbers remain unchanged. The breaking of the chocolate can be solved, but it can also evolve among the ideas for example that a class of 20 persons slept in rooms during the excursion where there were 4-4 pieces of bunk-beds. How many rooms had to be rented? …And where will the class teacher sleep? Among the many different ideas there will be some where with the unchanged numerical and unchanged division task the whole part of the division result is a number by one bigger than the whole part, or the solution of the problem will be exactly the division remainder.

Interesting type of realistic word problems are the ones which can basically be solved not by arithmetical operation, but by logical inferences (certainly the arithmetical operations will also get a role in certain steps).

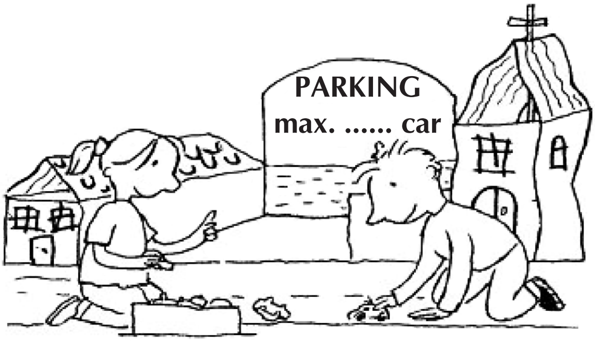
There is a bus stop in front of Eva's house from where the bus departs towards the school every 10 minutes between 6 and 9 o'clock a.m. The journey takes 15 minutes. Éva must be at the school by 7:45. By what time should she be at the bus stop in order not to be late for the school?

The can be different tasks found in the international literature which belong to the authentic category and which are appropriate for children belonging to the age group of grade 5-6. In an experiment made with 10-12 year old children there were several problems which stimulated the learners for activities as a result of which they can find the correct mathematical model to the realistic problem – often as a result of group work work.

In a well-known Flemish development program Verschaffel et al. applied for example the following example:

Pete and Annie build a miniature town with cardboard. The space between the church and the town hall seems the perfect location for a big parking lot. The available space has the format of a square with a side of 50 cm and is surrounded by walls except for its street side. Pete has already made a cardboard square of the appropriate size. What will be the maximum capacity of their parking lot?

1. *Fill in the maximum capacity of the parking lot on the banner.*
2. *Draw on the cardboard square how you can best divide the parking lot in parking spaces.*
3. *Explain how you came to your plan for the parking lot.*



All the typical characteristics of the authentic problems listed in our theoretical introductory chapter are met in the present problem:

- The picture belongs to the detailed presentation of the problem situation. Besides this a narrative story is outlined which together with the picture helps that the children feel the problem of their own that is they compare it with their previous experiences.
- The described situation has to be formulated by a genuine mathematical model. Based on the drawing and of the specified (and searched) data several different geometrical models can presumably be made.
- The learners have to obtain the other missing data. They can collect the missing data by field measurement, or by conversations.
- The complete problem is divided into several sub-problems: the setting of the different sub-problems, the checking of the attainment of the sub-aims is the duties of the learners.

In another very well-known intervention program Kramarski and Mevarech created the famous “pizza-task”. In this authentic task the prices of three pizza restaurants are given: the pizza’s diameter is given in centimeter (regarding the need to take into account the area of the circle the task is suggested to be used rather from the 7th grade) and the prices of the different pizza supplements are very varied. The student’s task is to find the best buy which also proves the above described characteristics of the task: verbal emulation of real situation, model-making, it should be decided which numbers are significant and which not, the task can be divided into sub-tasks, the solution process can be divided into sub-purposes.

The task mentioned in the previous part where we calculated the time intervals of the bus going to the school can be changed into an authentic task if the children look for the appropriate mathematical description according to their own, realistic, experienced travel habits.

The authentic tasks have a special role in assessing mathematical knowledge. We have seen that in the case of realistic tasks it not only the question whether "the end result will be found". The authentic tasks do not have an end result in the sense as the routine tasks have. But there is a solution process which is based on the comprehension of the text, on cooperative learning, on mathematical model making and the decision about the lack of data or their redundancy bring the learners into decision-making situation. By the end of the lower grades the sensitivity to problems, the conscious knowledge about and control of the phases of the problem-solving process can develop instead of the often rooted mathematical beliefs (for example, which task has a correct solution).

Similar to the routine word problems and to the realistic word problems in general the authentic word problems also offer possibilities for the use of a "reverse" problem-solving strategy: the creation of the problem situation and the text to a given mathematical structure. In an intervention program we have used with success already with 4th graders for example the task where they had to create a text to the division of 100:8 where the solution of the problem first should be division without remainder, in the second case division with remainder, in the third case the remainder, in the fourth case a whole number by one bigger than the whole part received by the division with remainder. By setting a problem of this type we clearly evaluate also creativity and verbal abilities which are not so much inherently connected to mathematics. This however cannot be criticized if we make it clear that we are diagnosing the application of mathematical knowledge in authentic problem situation.

Relations, Functions

The most important characteristic of the realistic problem types is that experiences of everyday life, in certain cases the specific knowledge get relevant role in the solution of the problems. In the case of certain tasks, it can be presumed that the correct solution requires the active utilization of everyday knowledge and experiences at least on one point of the problem solution (in the planning, implementation, or control phase). All this does not mean that the problem describes an everyday situation for the learner, the situation can be known, but a little strange to him/her, belonging to the world of the "adults". Such are for example situations related to the household, baking-cooking, travelling, shopping, saving.

What do expressions 1,5%, 2,8%, 3,6% mean on a milk carton?

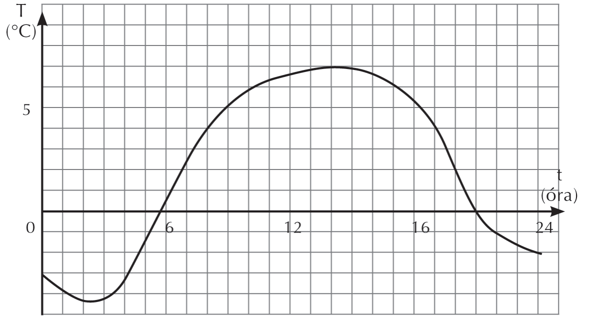
The proportion of margarine and farmer's cheese needed to make a scone is 4:5. How much farmer's cheese do we have to use if we add 20 dkg margarine to the pastry?

Is there direct or inverse proportion between the following pairs of quantities?

- *the length of the side of a square and its area*
- *the length of growing wheat and growing period*
- *sides of a square of 120 cm^2*
- *time needed to cover a given distance*
- *the mass and price of the fruit purchased*

A 24 cm high candle burns down to the bottom in 4 hours. In how many minutes after the lighting of the candle will it shorten to 16 cm?

The diagram below shows temperature values measured on a winter day.



When was the coldest? What was the highest temperature? In which period did the temperature decrease?

Csaba went for an excursion. During the first 3 hours he walked at a steady 4 km/h speed then he had half an hour of rest. After the rest he continued walking 2 hours at a 3 km/h speed when he arrived at his destina-

tion. He had a rest for an hour and a half, and returned home at a 3 km/h speed without rest.
Represent Csaba's movement in a coordinate system. Answer the questions on the basis of the figure: How many kilometers did Csaba make? How long did the excursion last? At how many kilometer distance was he from the point of departure at the end of the 9th hour? …………

It can be a practical characteristic in the case of the authentic problems that the solution of the task supposes the initiation, problem setting by the learner. It is by all means necessary that the learner translates the problem into his/her own language, feel it as his/her own in certain respect and be able to imagine the given situation. In many cases this simplifies the mathematical content of the problem and the key to the problem solution is the completion of this transformation, the finding of the correct solution model.

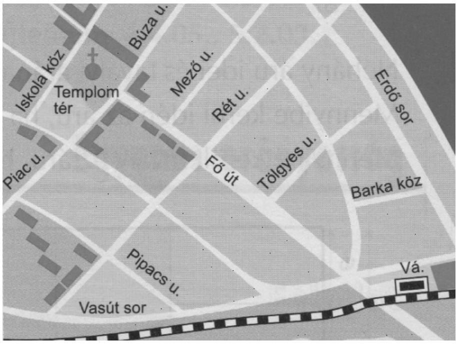
About 65% of the mass of a human body is water. How many kgs of water is in the body of a man of 80 kg? And in your body?

Is it really true that the discount on the ice-cream is more than 25%?

In the class the boys and girls are taking part in a steeplechase in separate groups. The boys made the 2 kilometers in one and half an hour, the girls made seven during 2 hours. Which team has won the speed competition on the 18 km distance?

At the parent-teacher meeting your mother would like to sit exactly on your place. Prepare a description, a "map" for her so that she could for sure find your place.

You can see a detailed map in the figure. 1 cm on the map is 20,000 times bigger in reality. How far is the church from the railway station?



In the more complex, unusual tasks we can direct students' thinking by questions and sub-tasks. During the development this method serves for the more detailed analysis of certain problems, for the recognition of relations, links, for the finding of different solutions. When evaluating students' work, the divergent solutions would make a problem, therefore it is worth to somehow determine the path of thinking.

At a telephone company we can phone two minutes for 80 Ft. If the conversation lasts longer they bill another 80 Ft and so on for every commenced 2 minutes. How does the fee of the conversation depend on the length of the call? Make a table and plot the cost per minute on a diagram.
Calculate the price of a 7 minute conversation. And that of a 12 minute call? Make a table and a diagram showing the total costs of the conversation.
Another company uses billing on a second basis. During a phone call we pay 1 Ft after every second passed. This company also charges fee for the calls, 30 Forint for each call.
The services of which company are cheaper?

Geometry

Constructions

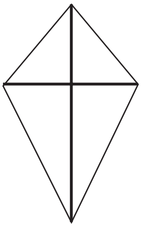
An example task on the properties of rectangles:

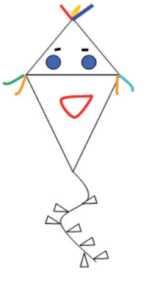
Uncle Robi would like to cover a parallelogram shaped area next to the sand pit of the children with 3,4 m by 3,6 m wooden plates. For this purpose he received congruent, symmetrical trapezoid shaped waste sheets of wood from the joinery in the street. The legs of the trapezoid are 6 cm long, the shorter base is 14 cm, and the longer one is 6 cm longer.
Will he be able to cover the parallelogram shaped area with trapezoid sheets?
How many sheets of wood does he need?

In this topic the authentic tasks require activities, where design, the follow up of the process and its planned checking and discussion get an accent.

Prepare a deltoid shaped kite. The symmetrical diagonal is 60 cm, the other diagonal is 40 cm. Design your kite. Measure how many laths, and how much paper you need. (You also need glue to finish it and rope to make it fly.)

You can also paint, decorate your kite.





Measurements

It is the task of the lower grades to lay the basis of length and area measurements. In grade 5 we are repeating and supplementing the acquired knowledge with the formula of the area of rectangular and square and with the formula of the volume of cuboid and cube.

Among the length measurement tasks the realistic problems require from the students to use their own experiences, or perhaps to follow trains of thoughts requiring occasional measuring tools.

Perimeter and area of a rectangle

Grandma has planted carrots in half of the rectangle shaped vegetble garden, radishes into ¼ of the land and spinnach in the remaining area. One side of the garden is 5 m, the other side is 8 m long.
Calculate how many m^2 area is occupied by spinach in the vegetable garden?
We had to fence the vegetable garden around with a low fence because of our dog named Buksi.
How many meters of fence do we need?

Volume of cuboid, unit conversion

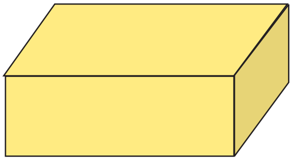
On our terrace there are 6 cuboid shape flower boxes. Their dimensions are: 100 cm by 30 cm by 40 cm.
How many m^3 potting soil do we need if we fully fill all the boxes?
Potting soil is sold in strong plastic bags. We bought soil in 50 liter bags.
How many bags do we need to fill the boxes?

Volume calculation, unit conversion

On April 20, 2010 a British oil production platform exploded in the Gulf of Mexico. During one week 795 thousand liter crude oil leaked into the sea. The spreading oil covered about 5000 km^2 surface. This accident is a serious harm to the environment.
Calculate how thick the oil slick can be if we consider the surface covered by oil of rectangular shape?

The problems based on the everyday situations children may experience mainly involve tasks in connection with cuboids, rectangles expecting active participation of the children.

In the school every child has a shoe box where the tools they need on drawing and mathematics classes are stored (paint, brushes, cups, cloth, and ruler, compass). The boxes should be put on top the each other so that they could be placed on the shelf. It was Klári's idea that everybody should cover the boxes by wall-paper so that they could look much nicer. Attila promised to buy the material. Wall-papers are sold in 10,05 m rolls with 0,53 m width.
Before making the calculations, give an estimatate whether this much of wall-paper would be enough for all the boxes if 24 children are learning in the class?
The size of a shoe box: 10 cm by 20 cm by 30 cm.

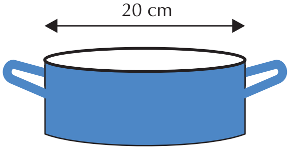


One of the taps is broken in the school's lavatory. Children measured how much water is wasted in 1 minute. They collected 60 drops of water into a measuring cup during 1 minute. The volume of dripped water was 13 ml.
Calculate how many liter of water is dripping away in 1 hour, 1 day, 30 days.
Estimate how many times one can take a shower with the water which is wasted during 30 days? Make calculation. (We generally consume about 75 l water when taking a shower.)

The following knowledge and skill elements can be diagnosed by this task: direct proportionality, unit conversions (time, volume units), computation skills (multiplication, division), and abilities for giving an estimation.

Volume calculation, unit conversion

Mix a little red pepper in 1 ml of cooking oil. Fill up with water a 20 cm diameter pot. Pour the painted oil on top of the water and observe the spreading of the oil. (The surface area of the water in the 20 cm diameter pot is about 314 cm^2.)
Calculate the thickness of the oil covering the water surface.

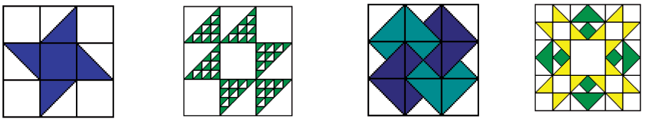


Transformations
Of the congruent transformations learners should know reflection around the axis and the shapes with axial symmetry (triangles, squares), as well as their construction.

The problems also contain enlargement, reduction tasks and the ratio of similarity is also present in the problems.

Axial symmetry

The favourite hobby of Aunt Zsóka is patchmaking. She made cushions decorated with patchwork for everybody as Christmas presents. Select the patterns with axial symmetry. Draw the symmetry axis.



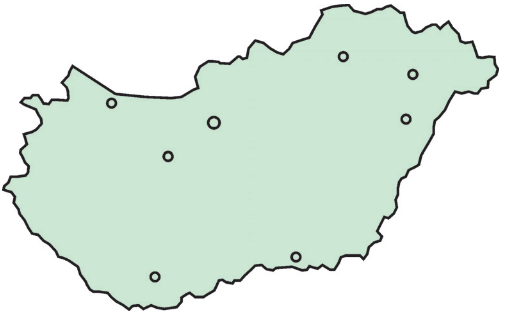
Reduction, proportion, area

The drawing shows the reduced floor plan of a medieval castle surrounded by a castle-ditch. One square side in reality corresponds to 10 meter.
About how many square meters could the ground surface of the castle be? Make a calculation.

Design patterns with axial symmetry and without it on a grid paper.

Similarity, enlargement, reduction, proportion

I have read on the internet that a miniature copy of Hungary would be built in Jakabszállás. It will be called Hungarian Garden and will occupy 93 thousand square meters. This area is about equal to the territory of 13 football fields. Check the territory of Hungary.
Calculate how many times the territory of Hungarian Garden is smaller than that of Hungary.
Enlarge the map on the picture to its double.



Orientation handled as separate geometrical field in grades 1-4 is not accentuated here. We present a sample task which can be regarded an extension of the assessment requirements of orientation in the lower grades, on the other hand it can also be listed to the authentic use of data pairs in the functions, relations topic.

Number the rows and columns in the classroom. Thus each seat is assigned a pair of numbers, where the first number shows the row, the second number shows the column.
Where do you sit?
Where is your desk neighbour sitting?
Who is sitting on chair (4; 4)?
Write down the index numbers of the boys.
Write down the numbers indicating girls in the second column.
Write down the index numbers of children with brown hair.
Write down the index numbers of children with blue eyes.

Combinatorics, Probability Calculation, Statistics

The most important feature of the realistic problems is that relevant role is given in the problem solutions to everyday life experiences, in certain cases to the specific knowledge. Starting from regular routine tasks we can for example arrive at realistic tasks if we attach characteristics to the actors, activities which have an impact on the possibilities to be taken into account during the solution.

A typical realistic problem:

Anna, Béla and Cili are siblings. Their parents ask them to do two sorts of housework every day: emptying the garbage can and watering the flowers. Make a plan that would split the chores between the three children in a fair way. In about how many days would the same child do the same housework?

In the field of combinatorics the examples requiring the computation of all the possible colouring versions of flags and maps are typical routine tasks.

The flags consisting of three different colour stripes are called tricolors, like for example the Hungarian or the French flags.

The stripes can be horizontal or vertical. How many types of tricolors can be prepared by using the red, white and blue colours? Which of them are actual flags of nations?

In the category of realistic problems the problems to be solved by using the pigeon-hole principle represent a typical class. In this age group we do not expect from the student the knowledge of the pigeon-hole principle in general, nevertheless the successful solution of the problem can be expected in case of familiar problems which can be modelled easily. In intuitively developing the principle we can proceed from the lower numbers to the numbers up to million.

In the class there are seven boys and all of them threw once with the dice after each other. Is it true that there will be at least two of them who threw the same number?
In one of the classes there are 20 pupils. How do we know for sure that there are some who were born in the same month?
The gymnastics teacher writesthe results of the small ball throwing contest rounded to meter. Why should it be certain that among the 200 senior class students of the school, there were some who achieved the same result in small ball throwing?

Regarding the topic of descriptive statistics the learners have a lot of opportunities for searching for models related to their everyday experiences.

In the interpretation of authenticity we follow the basic standpoint described in the introduction: we regard authentic a mathematical (word) problem if it describes a situation which can be regarded by the student realistic. In addition to the individual, paper-and-pencil type assessment methodology the demonstration of the reality of the problem situation often requires the generation of a problem context: everyday objects, text details, tables, etc. can be used as annexes to the problem text and the solution of the task is often made in group work. In the case of the paper-and-pencil assessment methods, or when using the individualized on-line diagnostic assessment method the surface characteristic of authentic problems is the longer, typographically varied and novel problem wording, while from deep-structural point of view as to the solution of the problem intransparency is a typical feature, that is the missing of an immediately applicable procedure leading to the solution. The authentic character of the task can be determined in a given historical-social environment, taking into account the majority of students belonging to a given age cohort. It is possible that for certain learners (or in other historical-cultural situation) an authentic problem changes into a routine task, what's more it may also happen that an authentic problem would not be regarded a routine mathematical problem from the point of view of certain learners or of certain historical-social context, but would be linked for example, to the assessment of critical thinking or of some kind of literacy domain.

A practical characteristic of the authentic problems is that the solution of the problem presumes the students' initiation. We can also say that in many cases the task of the learner is to create a mathematical task in the given problem sphere which requires the lower level application of the mathematical knowledge.

In the field of combinatorics the authentic problems require from the learners to recognize that a given everyday problem can be solved with the counting of the possible cases. A usual type of questions is the heap of problems starting with „How many different choices do I have?".

The authentic version of the realistic problem presented earlier is the following:

Anna, Béla and Cili are siblings. Their parents ask them to do two sorts of housework every day: emptying the garbage can and watering the flowers. Make a plan which would split the chores between the three children in a fair way. What other data would you collect about the brothers and sisters and the housework, based on which you can prepare the best division of work? (E.g. age, difficulty or period of the housework, etc.)

In the field of probability calculation the authentic problems contain the description of activities related to the everyday experiences of the learners: lottery, tossing-up a coin, sports games, card games. A usual type of authentic problems can be the problems based on the question „When do I have the biggest chance?". In the case of authentic problems belonging to the scope of probability calculation often linguistic-logical or game theory considerations lead to the solution.

Karcsi and Peti are throwing a rubber ball at a target in the middle of which there is a square with 20 centimeter sides. This square is just in the middle of another square with 30 centimeter sides the part of which outside the inner square makes the outer part of the target. Make a drawing about this peculiar target. One of the competitors has to hit the inner square, the other the outer part of the target. Karcsi has the choice which part of the target he wants to hit.
What do you suggest?

A student's answer should contain the area calculation data belonging to the task description (according to this the outer part is bigger). At the same time the authenticity of the task allows to make other questions: how are the hits counted – are scores taken away if the ball lands on the wrong ground?

Is it true that if somebody tries to hit a certain point, his throws will more frequently go close to that point than further from it?

In the field of statistics the authentic problems expect from the learners the ability to plan and to implement some kind of data collection process. They should be able to formulate questions about the data of some kind of property, to represent data with the corresponding methods: column chart, pie and scatter plot.

Based on the idea of the American NCTM Standards consider the following problem:

Compare which paper plane flies further: the one which is made of a soft copy paper, or the one which is made of hard carton paper of the same size. Make both planes with the same folding technology.

This problem requires the planning of the data collection process. How many throws should we make in order to get a reasonable result? Where should we make the experiment? With what devices and what precision should we measure the distances? How should the data be represented? Based on what shall we make decision and give an answer to the question?

It can be seen that this last problem practically encompasses all the features of the authentic problems (individual setting of problem, making mathematical models rooted in the everyday life, several possible outcomes, and cooperative mathematical activity). It should be noted that these types of problems are time-consuming, in average conditions they can use half of the school lesson. A lot of signs show, however that what we loose on the swings, we can gain on the roundabouts: the time-consuming authentic problems can primarily take the resources from the drill flavour routine examples.

Content Areas of Mathematical Knowledge in the Diagnostic Assessment

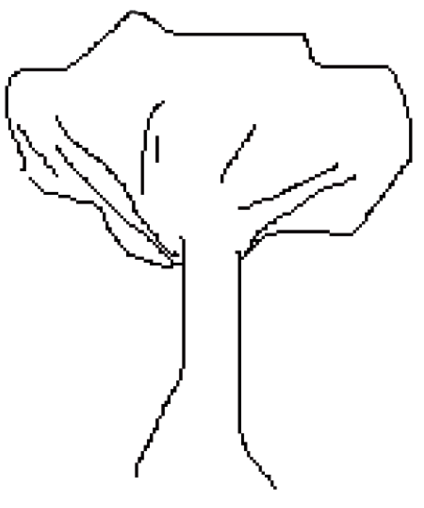
Detailed Assessment Frameworks of Grades 1-2

Numbers, Operations, Algebra

Numbers

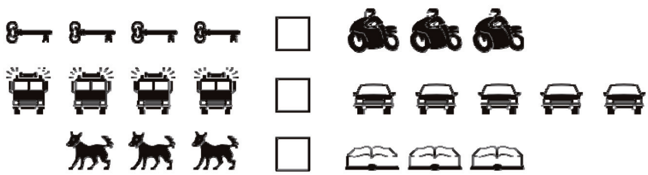
The development of number concept is closely related to "the same" relation. The understanding of the terms like more, less, the same, the learning and recognition of the different symbols (<; >; =), their filling up with correct content are basically assisted by the various pairing activities – using objects, pictures, drawings, words. In the course of the many different activities the learners gradually ascertain that if two groups, sets contain the same number of elements, their number is equal, that is – they are characterized by the same number, the same number relates to them – the number of the objects, things, living beings, etc. is the *same number of pieces.* The safe understanding, knowing of this relation, is a condition of the development of the correct number concept.

Draw three apples and two pears under the tree.



The problem focuses on the comparison of the number of pieces. The pupil can count, arrange in pairs, group the objects. The insurance of a lot of real experiences, the relating of number symbols to pictures, the use of number cards are very important. The writing of numbers (number symbols) can begin later.

Problem: On which side of the small squares do you see more objects? When you made a decision put the correct symbols (<; >;=) into the squares.



Problem: Which number is bigger? Put the < or > symbols between the numbers.

a) 8 – 2 9 – 1

b) 8 – 1 5 + 0

The development of the number concept is assisted by the empirical acquaintance with measurement index numbers. In grades one-two – as a continuation of the preschool preparation – the intelligent planning and direction of a great number of games (for example, filling space with cubes, with jug water, sand, bean, wheat, peas with the help of a glass, pouring them together, etc.) results that the pupils become capable of making comparisons (for example more, less, how many times the same) and besides the piece number they acquire the correct using of measurement index numbers. The experiences shall be understood, like for example: (1) to fill jug of the same size you need to pour more times with a smaller cup and less times with a bigger cup; (2) the same length can be made of more smaller units and of less bigger units; (3) using the same units the heavier objects can be balanced by more units, while the lighter objects by less units. In these measurements we can select the units freely and we can also select the official units without demanding their knowledge.

When comparing actual masses we use the traditional twin-pan balance, which demonstrate equalities, inequalities in an excellent way. (This experience will leave such imprints, memories in children what can be used by us later during the teaching of the principle of balance.)

In the case of measurements the varied selection of units (for example, use of colourful rods) supports the creation of more general, safe basis of the number concept. The many different measurement experiences contribute to the preparation of the idea of proportional changes.

The children often put their toys in order. The order and the ordinal number of the figures may change. This momentum includes the recognition that the ordinal number is not fixed to a figure but it depends on how we line up the figures and from where their numbering is begun. We will see that the number of figures will not change because of the order they are put in, or due to what direction the counting is started from. The many diversified arrangement of a given number of figures, the changing of the places of figures, the frequent repetition of the words first, second, third, … will promote the realization of the term of ordinal number, the understanding of the difference between number, ordinal number and will support the preparation of the concept of number line, too (for example, direction of the increase, decrease of ordinal numbers, looking for neighbouring numbers).

The assessment of the development level of the number concept is made by testing characteristics, properties observable by external experts. The basic condition of developing assessment is that we know the possibilities and process of the development of the appropriate level of knowledge and consequently of the diagnosis of the development difficulties.

By the end of the first grade children should learn the natural numbers at least until 20, and at least until 100 by the end of the second grade. This means that in this number circle the safe number concept shall be developed, number symbols should be learned and used appropriately in writing and reading. Further objectives can be listed as follows: neighbouring numbers, even or odd numbers, ordering according to size, their positions compared to each other (number line), division in different ways (for example, to the sum of tens and ones), rounding to tens (when buying in cash the sums in Forint to 5 or 10).

Some examples of the diagnostic assessment of the development of number concept:

Cross the numerals on the figure:

6 Z 9 F

4 12

M 3

7 ?

Cross the numerals on the figure:

6 Z 9 F

+ p 12

= M B 3

In these two grades activities for experiencing negative numbers (meeting with directional quantities (for example, warmer-colder; before, after 8 o'clock; to the right, to the left from me; etc.)), fraction numbers (cutting a whole into pieces, folding, etc.) also appear.

Zero is difficult for the small school children not only as a symbol, but the handling of zero as a number is also a big challenge. The following example is an illustration of the outstanding importance of zero as a numeral and zero as a number:

a) Which number is bigger? Circle it.

9 – 2 5 + 1

b) Which number is bigger? Circle it.

9 – 2 6 + 0

In these two grades we can already begin the preparation of the concept of number system and of the place-value system by the grouping of different objects, of smaller-bigger animated figures, which most often is made by tens, by the making the ten-hundred overstepping known. Being familiar with concrete numbers written in the decimal system, the knowledge of the concept of one and ten in the decimal number system is a minimal prerequisite knowledge.

Operations

The children learn the connecting role, the interpretation, and the use of parentheses also through examples (simple word problems, sums, taking away or multiplication of difference).

In the first two grades we expect skill-level verbal computation, addition and subtraction up to 20 together with the checking of results. In the first grade the breaking down of the learned numbers into the sum of two numbers, additions and the knowledge of adding three members is an expectation on practical level, while in the second grade this is a requirement up to 100 added by the safe knowledge of „little (multiplication) table". The „little table" means the table of multiplication and inclusion up to hundred.

While in the first grade the children get some kind of routine in complementing operations with missing members, in solving open sentences, and in the checking the truth of statements, in the second grade they make open sentences containing even two variables not only true, but also „not true". They formulate statements and find out if they are true.

Algebra

In the first two grades the introduction of symbols, their verbal expression and marking in writing in different relations, connections (for example, open sentences) can be regarded as basic elements of preparation of algebra. Below is an example of this.

Select natural numbers smaller than 20 which make the following open sentences true.

$13 + \square = 18$	Solution: $\square = 5$
$30 + \triangle + \triangle < 40$	Solution: $\triangle = 0, 1, 2, 3, 4$

In these types of examples the same symbols represent the same numbers, but different symbols can mark not only different numbers.

For example: *both* $\square = 3$, $\triangle = 3$ *number pairs are solutions of the* $\triangle + \square = 6$ *open sentence.*

Word problems which can be solved by one or two arithmetical operations, or where the understanding of the problem is mainly proved by a written open sentence represent an important field of the use of algebraic symbols in the school.

In the first grade the simpler word problems can be solved by the addition or subtraction of two data. In these types of problems it is not necessary to introduce a symbol for the unknown. The introduction of a symbol has a meaning if in the problem one of the member of the sum, or either the minuend or the subtrahend is unknown. The meaning of the symbols should be confirmed either verbally or in writing already in the case of these simple examples.

We can give simple word problems already at this age through the careful interpretation of which the learner can get rid of a lot of unnecessary work. Such is for example the next task.

Which number is bigger than 17, but smaller than 13?

Solution: *There is no such number.* (If we ask them to mark the partial solutions on the number line it becomes clear that there is no number which meets both conditions at the same time.)

These types of tasks first help to understand the problem instead of first trying to select the operation or to give an answer.

Putting down in writing, making models for the learned numerals, operational symbols, relation symbols, unknown symbols and later the parentheses requires rather high level of abstraction from the small child. The discussion of the relations discovered by the pupils – and their communication methods – contribute to manifold cognitive processes. A picture, a text can be approached from many different directions, they can evoke different thoughts, and the results of the thinking process can be appropriately shown in many different ways.

On Mother's Day Luci gave a bouquet of wild flowers to her mother. It contained 15 blow-ball flowers and by 10 more poppy flowers. Of how many flowers was the bouquet made of?

Solution: $15 + (15 + 10) = \triangle$, $\triangle = 40$; *there were 40 flowers in the bouquet.*

(In this case parenthesis means the coherence, but can be left, too.)

Formulate the following number problem by words. Write a word problem, too.

$4 \times (65\ Ft + 35\ Ft) = \triangle\ Ft$

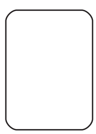
Solution for example: *The breakfast of our 4 member family was yoghurt for 65 Ft and a cheese biscuit for 35 Ft each. How much did a family breakfast cost?*

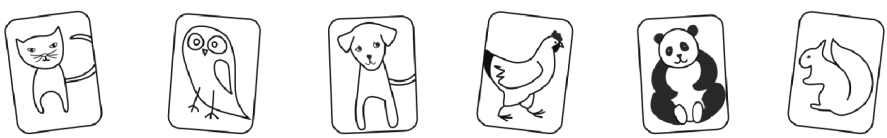
The writing down of word problems by symbols, making texts to the different notes, that is the frequent and factual practicing of the „to-and-back route" deepen the understanding of the content of concepts, makes the learner able to formulate in mathematical language simple word problems or to create adequate, simple texts to mathematical symbols.

A significant part of word problems relates to the open sentences. The verbal formulation, writing down of the open sentence based on a given text, making the contained unknown(s) concrete, or to replace them by actual elements can make the thus produced statement true or false. The scope of interpretation of the majority of open sentences is limited to the elements of the learned set of numbers, but elements of the basic set can be selected from many other fields, let's say from the world of flora and fauna, from the world of tales, too.

In the next example we put cards with pictures of different animals on the table. The cards show the picture of a domestic animal or a wild animal. During the solution the selected cards should be actually put into the frame, and it should be stated after this if the decision was right. This envisages that it should be decided in advance about each element of the basic set if it is a solution or not.

From the cards below pick the ones which make the statement true.

On the cards put to  *- you can see a domestic animal.*



a) b) c) d) e) f)

During the solution of number problems the learners can collect experiences about the unnecessary use of parentheses, or about how their placing influences the result. The need for using parentheses should be demonstrated in connection with the word problems, too.

Aunt Juli buys 1 liter milk for 140 Ft and 1 kg bread for 160 Ft every day. How much money does she spend a week for milk and bread together?

Solution: *7×(140 + 160) Ft = 2100 Ft. Aunt Juli spent 2100 Ft for milk and bread a week.*

For two years Aunt Kati bought one bar of chocolate for each of her four cousins and for her three friends for their birthdays. How many bars of chocolate did she buy during this period if she bought chocolate only for these children?

Mark the letter of the correct solution.

a) $2 + 4 + 3$ *b)* $2 \times (4 + 3)$ *c)* $2 \times 4 + 3$ *d)* $2 \times 4 + 2 \times 3$ *e)* $(3 + 4) \times 2$

Solution: b), d), e).

The children will check the correctness of the answers by the calculation of the results of operations and by „experimenting" with the results. Some of them will calculate the result by deduction and will look for the operations giving this result, others – probably less – will select the operations giving the good solution without knowing the end result.

During the first two grades the children should be able to formulate statements about simple activities, pictures, drawings, they should make a decision about their trueness, and they should make open sentences true by addition, and close them by replacement.

In order to develop the problem solving strategies it is important to establish double-direction relations between the things and relations in the problem and between the mathematical steps leading to the solution. To this end the pupils should be able already in grades 1-2 to find the correct word problem (or task consisting of an image) to a given mathematical structure. By the end of these two grades they should be able to formulate individually and collectively number problems, open sentences based on various activities and simple texts. As it was shown above they should be able to pick the ones matching to the text from given solution possibilities, and vice versa, they should select the right text to number problems, open sentences, or to create simple, clearly formulated texts.

Of the texts below select the ones matching to the following open sentence.

3 + 37 + 28 + ☐ + ☐ = 100

a) Aunt Bori lives on a farm and raises a total of 100 poultries. She has three cocks, 37 hens and 28 ducks, and as many geese as turkeys. How many geese does Aunt Bori have?

b) Évike was collecting the crop under their walnut tree. She collected 3 walnuts on Monday, 37 on Tuesday, 28 walnuts on Wednesday, the same number of walnuts on Thursday and Friday; and on Saturday the whole day she collected 100 walnuts. On Sunday she didn't pick any but counted the nuts she picked earlier. How many walnuts did she collect during the week?

c) Aunt Kati baked five different types of cakes for the birthday of her grandchild. She made "Gerbaud" cake, nut cake, chocolate balls, sour cherry pie and apple strudel. She took 50 pcs or 50 slices of each to the party. At the end of the party she counted the remaining cakes and said: Exactly 100 cakes are left. As I see the chocolate balls were the most popular, only 3 are left. The pie and the strudel were consumed equally. The nut cake was the least popular, 37 were left and they had not eaten 28 Gerbaud cakes. How many slices of sour cherry pie was left?

Solution: *Text b) does not correspond to the open sentence.*

By the end of the 2nd grade the learners know that comprehension is the first and most important step to the solution of a word problem. The understanding is made easier by playing, displaying, drawing, if necessary by rational re-formulation; understanding is followed by writing down with a number problem or with an open sentence, sequences, table of a problem and by computation, looking for the rule and by checking, relating to the initial problem, by comparison with data, real life, by preliminary estimation, and finally by the formulation, writing down of the answer. During assessment the steps of solution of the word problem are divided into separate problem units which can be evaluated independently ensuring by this that the eventual computation mistakes do not make invaluable the other, in principle correct steps of the solution of the problem.

Relations, Functions

At the age of grades 1-2 the following development problems and assessment requirements occur in connection with the content basis of the subject: continuation of sequences and looking for rules in the case of sequences consisting of objects or drawing symbols. The pupils should be able to generate sequences on the basis of a given rule. They should formulate in words the regularity determining the sequences.

By the end of grades 1-2 the following requirements can be set in the field of data pairs and data triads. The pupils should be able to recognize the relations between the matching members of two sets and based on the recognized rule set up appropriate pairs. Objects, persons, words and numbers known from their environment can all be used as elements of sets to be matched. They should be able to mark by arrows the relations between numbers and quantities. They should be able to arrange the related number pairs in tables and to recognize and continue the rule („computer game") in the case of number pairs arranged in table. In this age group the rule expressing the relations between the number pairs can be a simple, linear rule, or can be related to the sum of numerals, or to the formal properties of numbers. They should be able to represent in the Cartesian coordinate system specific points defined by coherent data pairs.

In the most typical cases of number triads it is about numbers of basic computation and the results of doing operations. For example in a subtraction operation three numeral data can be found the place of which cannot be changed compared to the operational symbols. We can arrange in a "machine-game" type table the thus relating number triads.

In grades 1-2 the typical examples of relations and functions include the continuation, addition of sequences about which the rule was determined.

The elements of sequences can be

- simple geometrical forms, for example, ☐◆❍☐◆❍☐◆…
- numbers, for example, 1 3 5 7 …
- symbols from different content areas, for example, a á b …

By the end of the 2nd grade the pupils have to be able to recognize the rule of quotient sequences within the 10x10 multiplication table.

The table arrangement is the typical form of problems built of the relations between data pairs, where we expect the continuation of the table after the recognition of the rule. Similarly to the sequences the data pairs also can have mathematical content or they can be connected to other symbol systems, and within the mathematical content geometrical and arithmetical phenomena occur typically.

Continue filling in the table.

□	❍	◇	▢	
■	●	◆		◗

5	11	3	4	14
3	9	1		

g	t		c	
gy	ty	ly	cs	zs

By the end of the 2nd grade, symbols representing data series have to appear in the table arrangement of data pairs (for example, the symbol of one data series is △, of the other is □) and the rule should be formulated by abstract symbols.

What can be the rule in the following table? What should be done with number in row of △ so that we get the numbers below in the row of □?

△	3	4	6	7
□	8	10	14	

The solution which can be expected from the pupil can be the following: „I add one to the number in row △ and I multiply this number by two and thus I get the number in row □." or „I take the double of the number in row △, then I add 2 to this number and I get the number in row □."

Geometry

One of the characteristics of teaching geometry is the learning-by-doing activities in grades 1-2. The experiences and knowledge gained during the great variety of activities give the basis to the conceptual building work in the lower grades and in the later years. In this age the priority of activities with spatial forms is evident, since taking the forms in hand, touching them, feeling things in general by hand belong to the first experiences of getting acquainted with the surrounding world. For this reason the constructions composing one pillar of the geometrical requirements begin with the three-dimensional (spatial) forms. Children of preschool age are already able to select from the toys the one which we ask from them by the often mentioned and heard words (for example, Give me the red cube.). In this phase the words, names are working as associations closely related to specific objects; the definition of cube as abstract concept is not the result of conscious school development. The main point of development – especially in the lower grades – is the active and conscious learning-by-doing approach, making the children discover through specific activities, and the use of definitions (and words) consequently. Playing games is a rightful demand and expectation of children coming from the kindergarten. All textbooks taking into account the age specificities, the psychological and mental developments offer plays, smaller competitions, humorous tasks which are by all means necessary to the healthy development.

The geometrical requirements can basically be put into four big groups: constructions, transformations, orientation and measurement.

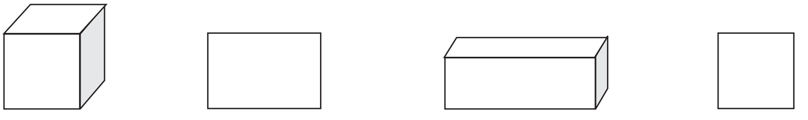
Constructions

Spatial and two-dimensional constructs and the studying of their properties are in the focus of this partial area of geometry.

We mainly examine the formal qualities of free works then of works connected to certain conditions and we lay the basis of the development of definitions. The series of varied, manipulative activities – cuttings, folding, gluing, copying to transparent paper, colouring, working with objects, drawings, building of cubes by adding new cubes or taking away cubes – mean the knowing and recognition of the properties of the finished forms. Children recognize identical and different characteristics and they formulate them in words with their vocabulary. The pupils will be able to identify the forms and to differentiate them based on the view or geometrical properties of forms; they are able to separate forms – setting simple, specific conditions – based on the geometrical properties.

On the basis of the general view they recognize the cube, cuboid, square, and rectangle. The filling with content of the geometrical system of symbols, the comprehension of relations develops gradually.

Name the forms on the picture.



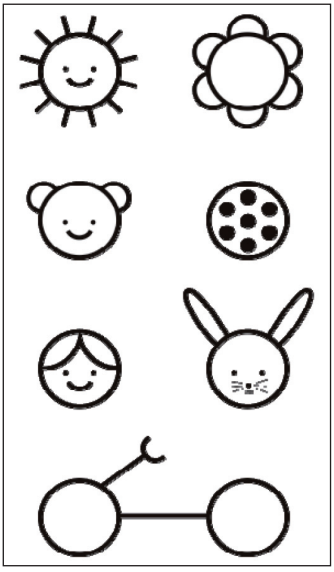
....

The more detailed discussion of the properties of bodies and plane surfaces (here squares, rectangles play greater role) generally comes in the second school year, where the conscious use of curved line, straight line, closing (closed) line, corner, sheet and edge, plus in given cases their numerical determination give help in the characterization of forms. After the detailed observation of forms we can build different bodies in simpler cases of specified layouts, and we can try the construction forms from the well-prepared silhouettes (projections).

1. *Put the cards in front of you containing a circle into separate groups.*
2. *Put those letter cards where the letters are only consisting of straight lines into different groups.*
3. *Take the box of colour rods. Put the red rod in the middle of the table, below that the smaller ones.* (Since the box contains several rods of every colour, we can give an additional instruction: „It is enough to put only one rod of one colour.")
4. *Build a decorative fence by using the colour rods. A castle with four towers. etc.*
5. *Build of colour rods the body on the table.* (First make cubes, rectangles and simple forms of them.)

We should choose tasks which create cheerful, playful atmosphere and at the same time serve a lot of different improvements.

1. *Supplement the big circle on the sheet freely so that we see a rabbit head. Colour it in.* (After visual checking we appreciate the ideas.)
2. *The circle can only be complemented by triangles so that at the end we can see a cat head.*
3. *You see circles on the paper in front of you. Supplement all of them freely.*



At the end of the work put the drawings on the big board and talk about them:

- What is common, what is different on the drawings?
- On how many pictures is there animal figure?
- Who made what of the second circle of the first row? And of the first circle of the third row?
- They should put questions in connection with the small exhibition.
- They tell true and false statements, opinions about the drawings.
- Which one do they like the best? Why?
- They have to try to find out what is shown on the different drawings.

The activities, constructions, observations similar to the above written make the pupils able to formulate short word problems themselves and to ask meaningful questions. The can put the same geometrical content into different „text robe", too.

Children can develop in many different fields simultaneously, if the work is well organized. It may happen, however that there is no parallelism between the development of mental and verbal abilities. We may think that the child is not answering because he/she does not know the answer, whereas he/she is not fast enough in the formulation, or his/her vocabulary is not wide enough and cannot find the proper words to give an answer. The word problems of mathematics and within this of geometry – be they only simple thoughts which can be expressed by some words – are efficient means of the development of comprehensive, explanatory reading, understanding and creating of texts.

As a result of the development of acceptable self-confidence and language environment the children will be able to use the mathematical vocabulary correctly and to make accurate and distinct verbal formulations.

Build the following two buildings with the white cubes of the colour rod set or with sugar cubes. You can see their floor plan here.

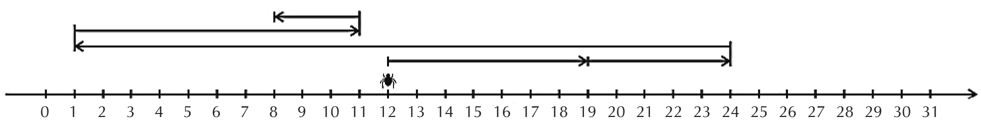
3	2	1
2	2	1
1	1	1

1	2	3
	2	3
		3

The numbers show how many small cubes should be put on top of each other.

Tools needed to the following task: a number line where whole numbers are indicated from 0 to 30.

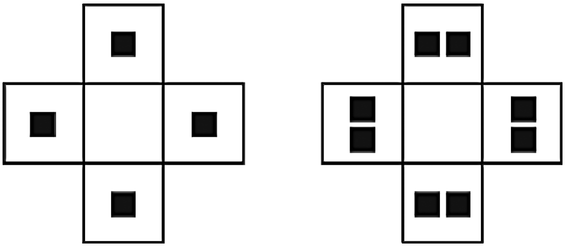
A flea is sitting on point 12 of the number line. Then suddenly, he starts jumping. First he jumps 7 units to the right, then again 5 units to the right and from here 23 units to the left, then again 10 units to the right, finally 3 units to the left. By now, how many units is he from the starting point?



The following can be a task for group-work:

Four children sit down on four sides of the table. They construct a collective building on the wrapping paper spread out on the table according to the given conditions. The building can be made of sugar cubes, white rods.

We draw a cross form of five big and congruent squares on a wrapping paper. The square in the middle remains empty, and on the four „protruding" squares we draw congruent plane squares in simple arrangement, for example as seen on the following figures.



Is it possible to create a construction which shows from four different directions exactly the view presented by the small black squares?

Solution:

1. Where only one-one small black square is drawn in each of the four bigger squares, the construction is made by only one cube placed well in the middle.

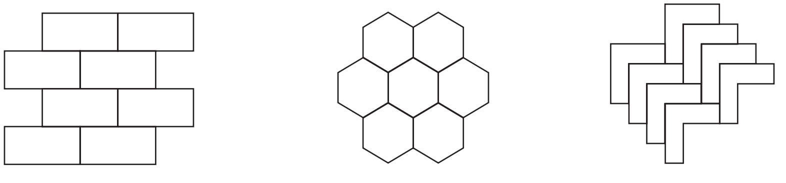
2. Where all the four projections are two squares next to each other the construction of several different forms gives a good solution. The diagonal placing of two cubes and the good placing of 3 and 4 cubes are equally good solutions.

The children in general put four cubes in the middle, in this case we suggest them to try to take away from them so that their side view should not change.

Already at this age we can give tasks which have several possible solutions and the learners should become able to find all the good solutions. Here we always think of simple tasks corresponding to the children's age.

In the second grade the geometrical activity covers the enclosing of different plane forms by thread, string, the complete (tight and one layer) covering of planes by different units. *These activities serve the practical preparation of the definitions of perimeter, area.*

Such coverings are the following:

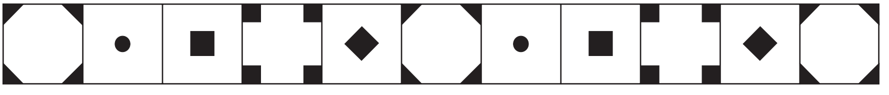


Transformations

Transformations include activities of moving the different drawings and shapes (reflection, translation, turning) and of the realization of the directions when the drawings or shapes move.

During the movement of the two- and three-dimensional drawings, in the course of moving them on different grids we can observe how the properties of the original and newly produced figures change compared to each other. Are there heritable (invariable) formal and dimensional properties? For example we enclose two adjacent squares on a quadratic grid, we copy this to another paper, cut it out precisely and move the received rectangle in a way that we determine a direction with the controlled connection of two grid points and we shift somewhat in this direction the cut rectangle. Or we glue the rectangle to a straw and mark a grid point around which we rotate the rectangle, or we fold the paper along the straight line connecting two grid points and mark the place of the reflection of the rectangle.

The geometrical transformation problems improve the creative fantasy, creativity, wit, aesthetical sense. We can produce beautiful line patters by reflection, translation. At this age children are already able to tell the difference between the reflected image and the shifted image based on the general view. It is very easy to make beautiful patterns by folding the small paper napkins precisely, make cutouts and cut ins in them. For example after folding the napkin twice we can cut out one of the corners, open it and the children can see the produced pattern. Encourage them to make different patterns. If we cut several napkins folded together, we can put on the board periodically changing their sequence (for example, the period has 5 elements) and we can ask a lot of questions, formulate many different word problems in connection with the beautiful view. (The black parts indicate the missing paper.)



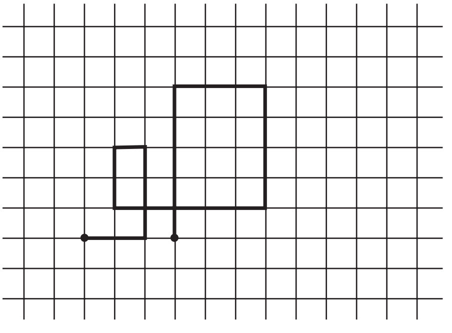
1. *Which pattern contains the least amount of paper?*
2. *How many holes are there on paper napkin number 5?*
3. *Do napkins number 3 and 7 have the same pattern? And of napkins numbers 3 and 8?*
4. *If we continued the pattern in the same way what type of patterns would napkins no. 20, 30 and 100 have?*

Orientation

The expression of position relations, showing the directions, finding places characterized by data will significantly contribute to the improvement of spatial orientation of the children, to the development of their correct orientation in space and plane. The elementary empirical level preparation of the definition of number line and later of the coordinate system (comparing the positions first by using terms like ahead, backward, under, above, next to, behind, farther, closer, between the two, by two to the right, by three to the left, and later expressing the relations by numbers, etc.) is supported by a series of activities related to this subject.

In the example below everybody has a squared paper, where the four cardinal points are indicated in the direction of the grid lines:

I start towards East from any grid point of a squared paper. I always follow the grid line and turn at the grid point. I always make a turn to the left. The length unit is the square side. The lengths of the routes taken are the following in sequence: 2, 3, 1, 2, 5, 4, 3, 5.
Draw my route.
How many units is the distance between the starting point and the end point of the route?



Solution: the starting point and the end point of the route is at 3 units distance from each other.

Measurements

The field of measurements consists of activities, requirements aiming at the examination of measurable geometrical properties.

The characteristics of spatial and plane objects contain – in addition to the formal properties – quantitative data which can be expressed by numbers. This group of activities and requirements can be connected to other fields of mathematics, too, for example it contributes to the development and strengthening of the number and operation concepts. Measuring length, perimeter, area, mass, volume, time should be made by many different optionally selected and by some standard units (for example, meter, kilogram, liter), the period of time, hour, day, week should be used correctly embedded in many different situations. Experiences gained during the well planned activities make possible the discovery of the relations between the units, quantity, index number, the empirical preparation of proportional changes. *By the end of grades 1-2 it can be expected that the pupil be proficient in practical measurements made with ad hoc units, should know the standard units and use them in practice.*

Fill in the missing numbers on the places of dots.

1 dm = ... cm	*6 cm + 2 dm = ... cm*
3 cm + 1 dm = ... cm	*2 dm – 15 cm = ... cm*
2 dm – 7 cm = ... cm	*.... cm + 1 dm = 12 cm*

The teaching of the four content domains within geometry requires a lot of tools. *During the assessment process, in addition to the written tests we also have to use manipulative and activity tasks in order to evaluate their practical solution, implementation.*

For the development and application of geometrical terms there are a lot of possibilities outside the school lessons, too. These are smaller tasks, projects which can be performed by the cooperation of the family, or friends setting shorter-longer deadlines and which in addition to extending the geometrical knowledge contribute to the improvement of division of labour between the member, to gaining shared feeling of achievement and to development of the other components of social competence.

In the field of measurement we can make important steps be putting into the children’s hand the meter rule, the measuring tape, the cubic measures, twin-pan balance with the proper weights and make a lot of measurements. We can prepare for example a bag into which we can put sand, gravel, bean, corn, wheat, small fruits, sandwiches in order to compare them, measure their weight. The unit can be a lot of things. Measure the time needed for the completion of different activities. We propose the use of round face clock supplied with 1-12 number and well recognizable hands.

The practical knowledge, use in specific tasks of standard units (m, dm, cm, kg, dkg; l, dl; hour, minute, day, week, month, year) is a requirement. This does not exclude the use of occasional units.

Examples:

1. *Kati measured the ribbons bought to the packaging of Christmas presents. Of the golden colour there was 15 meter, of the silver 100 decimeter, of the green 250 centimeter. Kati also bought red ribbon thus she exactly had the 35 m ribbon needed to packaging. How many decimeters is the red ribbon?*
2. *Arriving at the forest clearing, mother took out a bottle of juice. Each of her five children drank two full glasses of juice and the bottle became empty. They recognized only then that Dani's glass was twice as large as the identical glasses of the other four children. How many small glasses of juice were in the bottle?*
3. Measure the distance between two trees with different length units. Write down the index number and the unit. Based on the experiences make the following open sentence true: the smaller unit I measure the same distance, the bigger will be the …......................... .
4. *Fill up several plastic boxes of the same size with different materials. You can put sand, gravel, nails, lens, flour, etc. into them. With the help of a bascule compare their mass and put the boxes into increasing order on the basis of their masses. By writing down the order check their correctness by measuring with standard units.*

Put two questions during the measurements: What did you measure? What did you measure with? Can you show with your two hands how long 1 meter is? And 3 decimeter? And 5 cm? Can you put half kilo gravel on my plate? And 30 dekagram sand on this? Etc. And check every estimation by measuring and discussing the experiences. In this way we can achieve that the estimation skill of the pupil improves and that they become able to recognize the relation of units of different sizes.

The main point of measurement is comparison. At the beginning it is indifferent when, what and with what is the comparison made. For example we can use a piece of string, lath, the length of the step of pupils, the distance between two fingers of our stretched palm, etc. to measure length. When measuring volume we can use a cup or a paper glass of water, sand, bean, etc. We should select these so-called occasional units together with the pupils and should use in the task consistently.

Organize a competition consisting of 5-10 measurements. Before starting measurement estimate the probable result. Make a four column table on a sheet of paper where in the first column the measurement task, next to it the estimated result should be entered by the children, in the third column the actually measured value, while in the 4th column the difference of the estimated and measured values shall be written. The more times we organize such tasks the greater the probability is that the difference between the estimated and measured results will be small.

1. Select the odd one out.

a) cm m kg dm

b) minute year month dm hour

2. Let's play Twenty Questions game. (For example: the "logical set"[3]*, or the cards with different figures are in front of the children, one of them think of one of the papers or cards, the others ask questions about their characteristics (for example, is it hollow, not a triangle, red, not small, has a peak) and only yes or no answers can be given. After each answer everybody „screens" his/her own cards separately, that is they keep only those cards which can be a possible solution. The winner is the pupil who finds first the correct figure.)*

3 The "logical set" [logikai készlet] is a widely known and used set of coloured plastic plates with three types of shapes (circle, triangle, square), and half of the plates are holed.

Combinatorics, Probability Calculation, Statistics

During the development of combinatorial reasoning the teachers generally keep in mind the following stages:

a) Production of one of some cases meeting a given condition;
b) Production of as many as possible cases according to the given condition;
c) Finding all cases, ordering of the found cases and replacement of the shortages of the system;
d) Building of a system to the finding of cases of the given condition.

Of the above-listed four requirements a) and b) can be the means of developing induction reasoning. Assessing mathematical knowledge in a disciplinary sense we can mainly formulate c) and d) type requirements.

A typical problem in mathematical tests is the following:

How many two-digit numbers can you make of these number cards? Make all of them.

3 **4** **6** **2**

An equivalent formulation of the problem:

How many monogram can you make of these letter cards?

A **B** **C** **D**

We have to give assistance to the solution of word problems to our pupils so that they could find the model describing the problem well.

We present the following example of a combinatorics problem which at the same time measures students' systematization ability:

Funny Ferko has mixed the magnetic letters in the label "2. a osztály" (class 2.a).
This is what he made:

2. o asztály

How many letters are on the wrong place?

In connection with the next problem we show that rather different solution strategies may lead to the right solution. The equivalent use of the manipulative, image and conceptual level solution possibilities confirms the links between everyday experiences and mathematical concepts.

Anna, Béla and Cili organized a running competition. They passed the finish line in different points of time. How many ways could they pass the finish line?

One possible solution of the task is that the teacher invites 3 children in front of the class. Children sitting on their places instruct those standing out. They determine the possible orders since it is difficult to keep in head the cases already considered, it is obvious that they have to model the problem. For example in a way that they write the names on a piece of paper (the same name can be written on several pieces of paper) and solve the task by putting these paper here and there. Evidently such problems cannot be solved in written tests.

In the case of problems formulated during the assessment care should be taken that we present a good model with the help of which the problem will be understandable. We can start the writing down of cases what the children have to continue. In this way the task is limited to writing down as many cases as possible where the accent is on keeping the criteria.

Anna, Béla and Cili organized a running competition. They passed the finish line at different points of time. How many ways could they pass the finish line?
Continue the options.

A,B,C; A,C,B; B,A,C; ______; ______; ______;

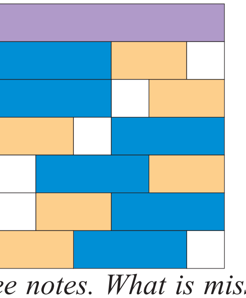
The other option is that we determine a system (for example a table), which makes the problem more clear-cut.

Anna, Béla and Cili organized running competition. They passed the finish line in different points of time. In how many ways could they pass the finish line?
Continue writing the options.

1.	A					
2.	B					
3.	C					

Already at this stage preparations are made for building up a system leading to finding all the cases belonging to the given conditions. For example examining the possible arrangements of the 3 elements it was observed in the classroom in how many ways the three stripe flag can be coloured using the red, white and green colours. Other examples:

1. In how many ways can the lilac rod be decomposed using different rods?



2. I prepared tunes of three notes. What is missing from the sequence?

do-mi-sol **mi-do-sol** **sol-mi-do**
do-sol-mi **mi-sol-do**

The task is getting more difficult when we increase the information noise.

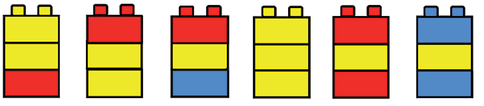
*3. Wh*at can be the last line of the „poem"?

Tirim taram turum
Tirim turum taram
Taram turum tirim
Taram tirim turum
Turum tirim taram
……………………

The structure of the flag colouring and the above three examples are the same, but their content is very different. In this age, however the structure is important for only very few children, that is why the different formulation of the same problem means a new challenge.

In the probability topic the separation of sure and not sure becomes important in the first two years of schooling. The problems formulated on the worksheets are preceded by many-many experiences.

I have built the following towers of red, yellow and blue Lego blocks. I have picked out one and made statements about the selected tower. Decide if the statement is for sure true.



	Surely true	Not sure that it's true
There is red block in it		
The middle element is yellow		
All three colours can be found in it		
There is no blue in it		
There are two identical elements		

Since children collected many everyday experiences about impossible events, we can make a try to ask about this difficult definition.

We put 5 red and 1 blue balls into a bag. Then we picked out two and made statements. Underline the statements which you think are false.

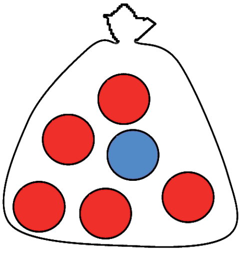
All of them are red

All of them are blue

There is blue among them

There is no red among them

There is blue among them



Detailed Assessment Frameworks of Grades 3-4

Numbers, Operations, Algebra

Numbers, set of numbers

Based on the reality content of numbers we extend the number concept up to 1000 in grade 3 and up to 10 000 in grade 4. Counting in the set of three and four-digit numbers has an important role, furthermore the estimation of piece numbers and measurement index numbers, the counting with approximation, and measuring with given precision with occasional and standard units becomes important in this age group. In the course of measuring practices the children will be able to define the relations expressed by measurements with different units, they will understand the conversion by units.

By using the different teaching tools the children acquire the command of numeral systems, gain experiences about grouping, conversion and exchange. The practical knowledge of the essential understanding of the decimal system and of the place value system makes for them the writing and reading of numbers safe, they will understand the system of numerals. Children are able to use reliably the formal, local and real values of numerals. They examine the numbers according to the familiar number properties or number relations (for example, parity, neighbouring numbers) and they get to know new number properties (for example, divisibility, number values rounded to tens, hundreds, thousands).

Circle the odd numbers.

1	2	4	5	6	8

Circle the neighbours of number two.

0	1	2	3	4	5

They recognize and are able to express the numbers in their different forms, they can judge the size of numbers and are able to put the numbers into increasing and decreasing order according to their size. They can place numbers on the number tables and on number lines with different scaling.

The children become acquainted with the concept of negative numbers in two interpretations. On the one hand negative numbers are interpreted as index numbers of vectored quantities (temperature, displacement, turning, time), on the other hand as deficits. To this end debt and asset cards are used. The numbers are compared by attaching specific content. The many different forms of the numbers are produced. They experience through activities that adding something does not always result increase of value, but taking away may result increase.

Word problems are often used to the assessment of computation ability. These problems which can be solved by one operation do not require data collection, they can be written down by a simple number problem and can be solved.

For example:
I had 750 Forint in my purse. I spent 480 Forint. How much money is left for me?
A bus ticket costs 320 Forint. What is the price of five bus tickets?

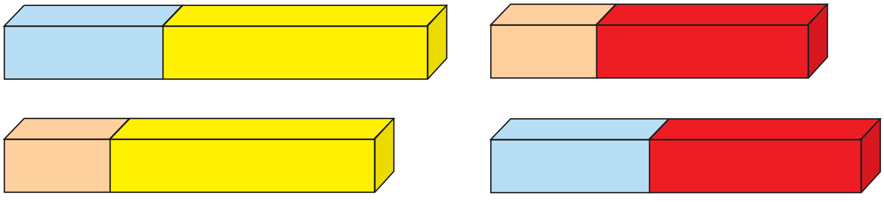
In the third grade children in many cases are computing with approximate values in the 1000 and in the fourth grade in the 10 000 number circle, the questions formulated by the problems also demand this.

For example:
Kati paid with a 1000 Forint banknote in the stationary shop. The cash-register showed 578 Ft.
How much is the return rounded to hundreds?

Operations

In the third and fourth grades the interpretation of the operations by objects, drawings, more abstracts figures and texts is necessary in the extended number circle, too. We pay special attention to the interpretation of operations with approximate numbers. The two-direction activities contribute to the understanding of mathematical models. On the one hand children read operations from displays, pictures, figures, on the other they collect examples to the given mathematical model, and they formulate problems. The interpretation of the addition, subtraction of bigger numbers is assisted by representation with segments or areas. This can be prepared by the use of colour rods.

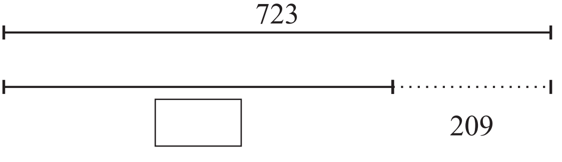
For example:
Let the white cube be worth 100. Which arrangement is close to the amount of 246 + 467?



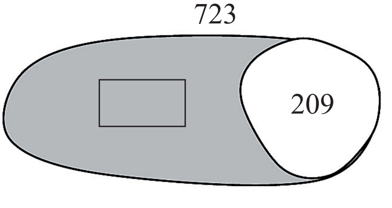
The settings, the readings about the settings can be followed by the use of segments, or areas. They are suitable for representing the numbers with approximations, for presenting the relations between the numbers.

For example:
One of the numbers is 723. This is bigger than the other by 209. What is the other number?

With segments:



With areas:

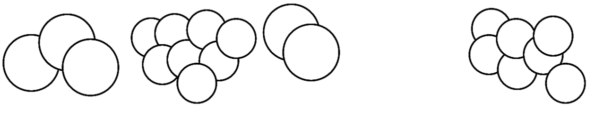


The verbal computation operations are made on the basis of analogies in the extended number circle. Their understanding is supported well by the use of tokens. The activities make the verbal computation ability of the children safe in the scope of round numbers. The computation procedures acquired by the children in the 100 system in the 2nd grade will be followed in the 3rd grade by round hundreds and tens in the 1000 number system, then in grade 4 by round thousands and hundreds in number system 10 000. In the computations the children use simplification procedures the basis of which is the unchangingness of the sum or of the difference. They can gain experiences about them by activities what they use during the computations.

How much is 380 + 270?

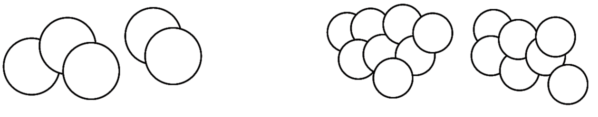
Presented by tokens:

Method 1: by division of the 2nd member:



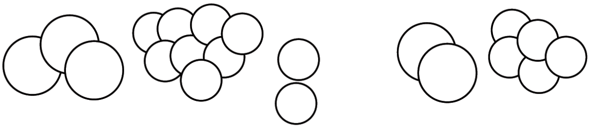
(380 + 200) + 70 = 580 + 70 = 650

Method 2: By adding the hundreds and tens:



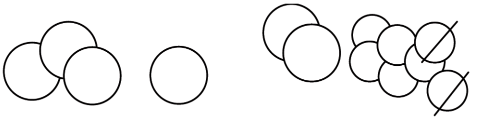
(300 + 200) + (80 + 70) = 500 + 150 = 650

Method 3: By placing from one member to the other:



(380 + 20) + (270 – 20) =400 + 250 = 650

Method 4: By increasing one member and decreasing the sum:



(400 + 270) – 20 = 670 – 20 = 650

The approximate calculations made with the rounded values will be necessary during the projection of the results of operations in writing.

The algorithms of written operations, the methods of checking the results of computations also build on the properties and relations of the operations. This also makes necessary the knowledge, purposeful use of the interchangeability or grouping of members, factors.

For example, in class 3, when the children have not yet learned multiplication in writing by two-digit numbers they are able to calculate the result of 26·24 by using multiplication in writing by one digit numbers. Some calculation options: (26 · 8) · 3 = (26 · 6) · 4 = (26 · 3) · 2 · 2 · 2. In grades 3-4 word problems get special importance in the development of problem solving ability. These are mostly complex tasks which cannot be solved directly. It is advisable to get to the solution of the problem by keeping some appropriate steps. The recognition of the problem is followed by its interpretation, by putting down the data and understanding their context. The children use many different models for the description of the relations between the known and unknown data. A number problem containing several operations can be a model for example. It is practical to use parenthesis in these descriptions even if you use them only for the indication of the coherence of data.

For example:

Peter's family organized a three days excursion by car. On the first day they travelled 160 km, on the second day 80 km more. On the third day by twice as much as on the first day. How many kilometers did Peter's family travel during three days?

Description of the relations of the problem by numbers:

160 + (160 + 80) + (160 · 2) =

Algebra

In addition to the word problems containing numerals, the open sentences appear and get greater emphasis. Continuing the activities began in grades 1 and 2 the finding of elements making the open sentences true or false is made by trial and error method, but the children also use the method of planned trial and error on finite basic sets in order to find the solution. Children are able to find (in case of simpler relations to create themselves) to the relations formulated between the known and unknown data the correct answer of the specified open sentences in the given situation.

I have thought of a number. I have subtracted 8 times of this number from 800 and received 12 times the number I thought of. What number did I think of?

With open sentence: $800 - \boxed{} \cdot 8 = \boxed{} \cdot 12$

The relations of problems – especially of the problems where the key words may be in contradiction with the arithmetic operation needed to be executed – are often written down in open sentences. For children of age group 8–10 it is often easier to recognize and write down operations to which the text refer, than reformulate it as inverse operation.

For example:
Csabi's school has 12 grades. The school has 160 pupils in the first four grades. These classes have twice as many children as the high school classes (grades 9 through 12). The number of children in the fist four grades is 40 more than the number of students in grades 5 through 8 combined. How many children study in the grades 5 through 8 and how many in the high school classes in Csabi's school?

We mark the number of high school children by: $\square$

The number of senior class children by: ∇

Using these symbols we can easily create the open sentences describing the problem:

$\square \cdot 2 = 160$ $\nabla + 40 = 160$

The solution of the word problems can be assisted by the use of sequence, tables, simplifying drawings or diagrams even in the case that the problem has only one solution.

For example, the solution of this problem can be found by the children by filling a table, too:

I have only 20 and 50 Forint coins in my purse, 12 coins in total. The value of the coins amount to 360 Ft. How many 20 and 50 forint coins do I have in my purse?

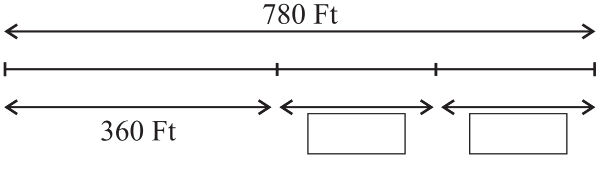
We can make to following table:

No. of 20 Ft coins	2	4	5	6	7	8
No. of 50 Ft coins	10	8	7	6	5	4
Value of 20 Ft coins	40	80	100	120	140	160
Value of 50 Ft coins	500	400	350	300	250	200
Total value of coins	540	480	450	420	390	**360**

A simplified drawing contributes to the solution of this problem:

In the stationary shop I bought two identical exercise-books and a pen and paid a total of 780 Forint. The price of the pen was 360 Forint. How much was one exercise-book?

The segment drawing can help to find the solution:



In the selected mathematical model of the problem the computations are followed by their checking. Checking can be made by comparison with the preliminary estimation, by inverse operation and we can use pocket calculator, too. If we select the inverse operation for checking this may confirm the relations between the operations.

Relations, Functions

The most important fields of development in the domain of relations, functions in grades 3-4 are:

- comparison, identification, ability of differentiation, observation;
- abilities of selection, sorting, systematizing and highlighting the importance;
- collection, recording, sorting of data;
- abstraction and materialization abilities;
- recognition of correlations, discovery of casual and other relationships, recognition, following of analogies;
- expression of experiences in different ways (by presenting, drawing, sorting of data, collection of examples, counter-examples, etc.), formulation by own vocabulary, in simpler cases by using mathematical language of symbol system.

Children are able to formulate in the language of mathematics the recognized relations, to express them by words, symbols, rules (in the case of function by arrow, in case of relations by open sentence). They are able to continue the commenced pairing according to the specified or recognized relation.

In grades 3-4, a new element in the treating of correlating data pairs is the graphical representation of relations in the Cartesian coordinate system. Since during the representation of data pairs the order of the members of the data pair is important, it is advisable to make exercises where we represent in a shared coordinate system data pairs produced by the exchange of the prefixes and suffixes.

The learners are able to arrange data, numbers in sequence according to their content or size, to formulate guesses as to the continuation. The express the recognized correlation by the continuation of the sequence or by words. They can continue the sequence on the basis of the formulated rule, they are able to check the compliance of the rule and the data. They look for different rules to the sequence started by some elements.

What can be the rule in the following table? What shall we do with numbers in the row of symbol △ in order to get the number in the row of symbol □?

△	3	4	6	7
□	8	10	14	

The solution expectable from the learner can be the following: „I add one to the number in row △, then I multiply this number by two and get the number in row □." In connection with this table it is possible to formulate a closed problem, where we ask the children to select the rule matching with the Table's data.

What can be the rule in the following table? Encircle the letter of the relation which is true for the table, and cross the one which is not true.

△	3	4	6	7
□	8	10	14	

a) □ = (△ + 1) · 2
b) □ = (△ – 1) · 2
c) □ = (△ + 2) + 3
d) □ = △ · 2 + 2

Geometry

In the field of geometry in grades 3 and 4 the same four sub-domains give the frameworks as in grades 1 and 2. The (1)constructions, (2) transformations, (3) orientations and (4) measurement domains cover all the learning objectives that we define in these grades in the field of geometry.

Construction

Similarly to grades 1-2 the requirements contain here, too the recognition and construction of cuboids, cube, rectangle and square. The learners learn the definitions of edge, base and lateral face.

The learners learn the expression of body mesh, specifically the typical two-dimensional nets of cuboids and cube.

Of the geometrical properties – during the practical activities – they learn the following terms: form, vicinity, direction, parallelism, perpendicularity.

The learner will become able to group bodies and plane figures on the basis of certain geometrical properties. Other typical characteristics observed while grouping objects: angularity, holedness, symmetry, identity and difference of dimensions.

The concept of reflection (symmetry) can be improved on the one hand by paper folding activities, on the other by building the reflected image of spatial forms.

Besides the priority of spatial forms the activities with plane drawings also get greater emphasis. Students will become able to copy bodies and plane figures, to create the reflective image of a plane figure or a body. The copying is primarily made by bodies, rods which can be taken in hand, but in grades 3-4 we increasingly utilize the abstraction possibilities offered by drawing.

The learners are able to use the compasses and the ruler. The basic level use of the compasses is implemented for example when the learner takes a 5 cm distance into the span of the compasses.

Transformations

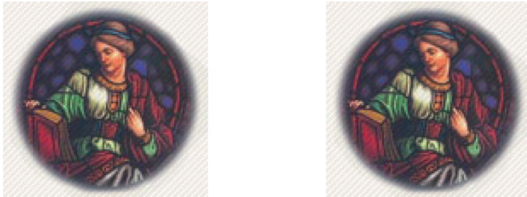
Built on the experiences acquired in grades 1-2 the manipulative and pictorial level components of the concept of congruence and similarity are developed. Students are able to recognize if two figures or their images are congruent or similar. They can confirm the identity or difference of formal features. In the case of the difference of figures they can formulate by words the type of difference (for example, longer, more oblique).

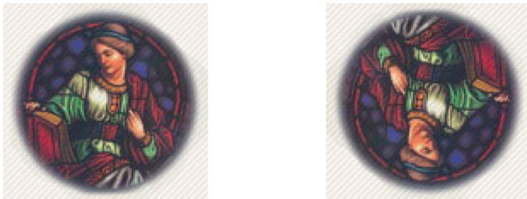
In the case of 3D shapes they are able to reduce or enlarge forms of the elements of the original body, on case of plane shapes with the help of the quadratic grid. Students can reflect plane figures along axis and rotate them with the help of a copying paper.

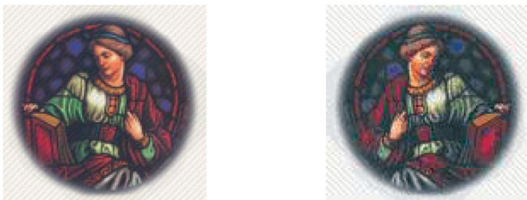
They can make difference between figures produced by translation and by reflection along axis, even in case of complex forms.

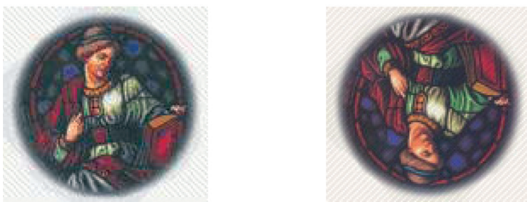
The next task evaluates the making of difference between reflection along axis and translation. The content of the problems is basically optional, there is no reference to (and there is no need for) the use everyday experiences.

In this example you have to decide about two matching figures if they can be transferred into each other by reflection along the axis or by translation. Write the letter of the figures in the corresponding row.

a) 

b) 

c) 

d) 

Can be transferred into each other by reflection along the axis: ....
Can be transferred by translation: ....

Orientation

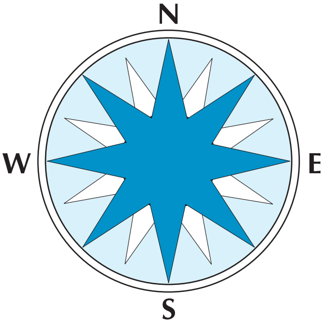
We have to mention that the major part of orientation in the geometrical sense is related to the orientation recognized as a category of geographical discipline. The connection of the two fields can be interpreted in a way that the development of orientation abilities is made in a different context. The orientation ability developed in mathematical context prepares the learning of the coordinate system as a universal mathematical means, during which we use definitions known from the everyday life. As we have indicated at the requirements of grades 1-2 the two, independent data typical in the case of the use of the plane coordinate system used as arranged data pairs give the basis of the orientation in the everyday meanings.

Orientation starts from the experiences collected during motions in the three dimensional environment. Learners of grades 3-4 are able to orientate on the basis of one, two or three data. Orientation on the basis of three data, which represents the mathematical model of spatial orientation is in practical life many times replaced by orientation on the basis of two data. The orientation ability of the learners include that they are able to receive and understand the relating information (for example, „if you step five ahead and two to the right you arrive at the destination") and they are themselves able to formulate the information needed to the orientation.

The construction of pictorial elements of orientation, for example the making of simple map drafts is discussed in another volume of this book series, in the geographical chapters of the science framework.

The next task might have even been included in the tests of the geographical or natural sciences subjects. In our opinion this does not question the validity of the task, since the context of the problem, the words of mathematics or natural sciences included in the title of the test make an influence on the performance of the learner. We consider it desirable that both the imaginary and verbal knowledge system creating the basis of orientation develop on good level both in mathematical and in other context.

On the figure you can see a compass rose where the four main cardinal directions are indicated. We go from the middle of the circle to the north. We turn back and go to the middle of the circle. Which cardinal direction is to the right from us in this case?



Measurement

Measurement is included in the subject of geometry in the Hungarian mathematical didactical traditions, while in the American „Principles and Standards for School Mathematics" which is regarded as an important reference basis for us, measurement appears as a separate chapter. The reason for this can partly be found in the different cultural traditions (for example, differences in the use of the metric system), it partly expresses our approach which considers measurement as an activity related to the well-known geometrical shapes. Since according to a much more general approach, which in the world of sciences is widely accepted, measurement is defined as the assignment of numbers to objects, events, properties according to a set of rules. Although there are efforts that this latter, general approach to the measurement of geometrical shapes also enter into the school (requirements of making measurements with the so-called „occasional units" in grades 1-2), the school practice is still characterized by the fast switch-over to the standard units, then by the immersion in the arithmetic operations of conversion of units.

In grades 3-4 the students should know the definitions of unit, quantity and index number. During the measurement activities the measurement of perimeter is made by enclosure, the measurement of area by overlapping, and the measurement of volume by occasional units („small cubes"). The subjects of perimeter, area and volume measurements should be rectangle in the case of plane shapes and cuboid in the case of bodies.

The learners should know the following units of measurements: mm, cm, dm, m, km, hl, l, dl, cl, ml, t, kg, g. They have to be able to convert into each other the "neighboring" units. The conversion should mainly be connected to practical activities, that is after the measurement made by one of the units we make a repetition measurement with the adjacent unit. In order to measure the time they have to know the hour, minute and second and to convert the neighbouring units into each other.

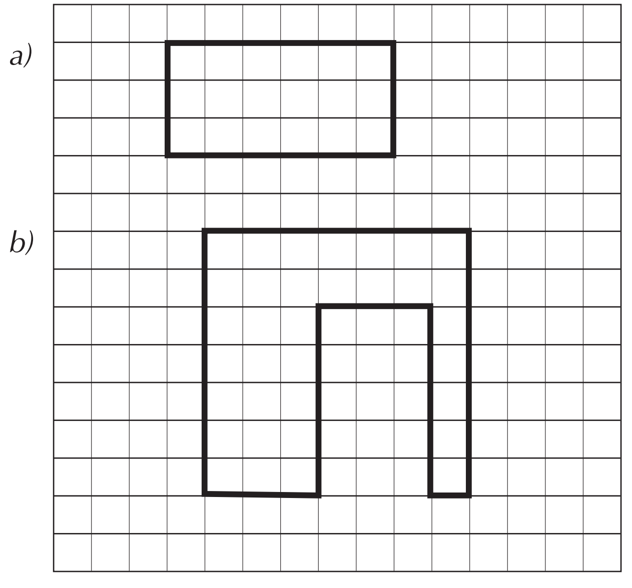
The greatest part of the practical exercises in connection with measurement is related to the conversion of units.

How many deciliters of milk was consumed today if we drank three one liter bottles of milk?
If the step of a child of grade 4 is 60 centimeters, how many steps does he need to make 12 meter?

In the radio, 5 minute long musical pieces were played for two hours. How many musical pieces were played during this time?
In England, mile is often used to measure distances. One mile is equal to 1 km plus 609 meters. How many meters is one mile?
The weight of a small box of butter is 100 gram. How many boxes should we buy if we want to buy 3 kg?

In addition to the conversion of units we can formulate simple perimeter and area calculation problems as world problems. From the manipulative activities of the children we get to the image level problem solution with the following tasks.

On the square grid table a small square means one unit area. Calculate the area of both plane figures combined framed by the bold line.



Area of plane figure *a)* is: ________

Area of plane figure *b)* is: ________

Combinatorics, Probability Calculation, Statistics

The teaching of combinatorics, probability calculation and statistics mainly aims at gaining of experiences also in grades 3-4.

Students' combinatorial reasoning is further on mainly shaped by making the importance of systematization understood. In grades 1-2 the children are not primarily interested in how many different possibilities are there, since it is the process of finding and producing different options that is important for them. In the case of production of a set of small number of elements the endeavour for completeness is already a realistic requirement in this age group. We have to continue to assist the children in finding an ordering principle, since this is important for finding all cases. We can give up providing stronghold to the commencement of the task in the case of very low number of elements.

The aim of the probability games in the classroom is to present that things which more often happened are more probable. In this case the teacher is a real participant of the experimentation of the learners and hopes that the outcome of the game will bring the „expected" result. In these years it can also be observed during the analysis of the games, that it can be considered more probable what may occur in different forms (even if this is not confirmed by the actual experimental data). Thus in the course of assessment the intuitive determination of smaller, or bigger probability is a requirement.

In all probability the terms "sure", "not sure", "probable", "possible" were built into the children's vocabulary during grades 1-2. The curriculum requirements formulate the separation of deterministic (sure or impossible) and non-deterministic (possible) events. Thus we can certainly ask only indirectly to what extent they consider the given event probable.

Observation, collection, recording, ordering of data appear in the curriculum requirements, too, which helps, besides the deeper understanding of statistical subject the making probability decisions.

In general the probability activities and tasks are not independent development targets, but are connected to other fields (for example, computation, geometry, and combinatorics). Let's say we throw with two dices and tips should be made about the parity of the multiplication. The solution of the problem requires from the children knowledge of the number theory, perhaps their computation ability, plus their ideas about probability. These should be taken into account during the preparation of test sheets.

The finding of the missing elements of an existing complete system is a different task for the searching of all cases, for the ordering of the found cases and for the addition of the deficit sin the system.

In this task the finding of all the missing elements can be a legitimate claim, since the pre-planned systems show the solution. Plus the task improves the orientation ability in the table.

The more simplified version of the above tasks can be:

Make 3 digit numbers from numbers 5, 2 and 7. Write down all the possible solutions.

At this grade the children regard important not how many possible options there are, but finding and producing the options are interesting for them. If in the activities they recognize the already tested similarities, they made a big step towards generalization. Consequently, in the combinatorics tasks (including their correction key) not only the number of all possible options is important, but the regularities manifested in the partial solutions, or in the listing of options can also be and should be evaluated.

Certainly, there is a possibility for the transformation to word problems of the formerly played and experienced activities. We still cannot give up providing help either in the form of a table or of a commenced tree-diagram. It is still important that the main question to be answered is not in how many different ways a thing can occur, but we should ask for the production of all the possible options from the children.

For example regarding the following two tasks the offering of a table solution seems effective.

Grandma is preparing to fill up her pantry, so she bought apples (A), pears (Pe) and plums (Pl) on the market. She would like to put the fruits on three shelves. She puts one sort of fruit on one shelf. In what order can she put the fruits?

Using the initial write all the possible solutions into the table indicating the shelves.

	Options					
Shelf III	Pl					
Shelf II	Pe					
Shelf I	A					

Put together a three-course menu (soup, main course, dessert) for the Falánk family so that none of the family members get the same three courses.

Menu

Soups:
- **m**eat soup
- **f**ruit soup

Main courses:
- **s**pagetti
- **r**oast a la Brasssó
- **p**otato casserole

Dessert:
- **d**umplings somlói

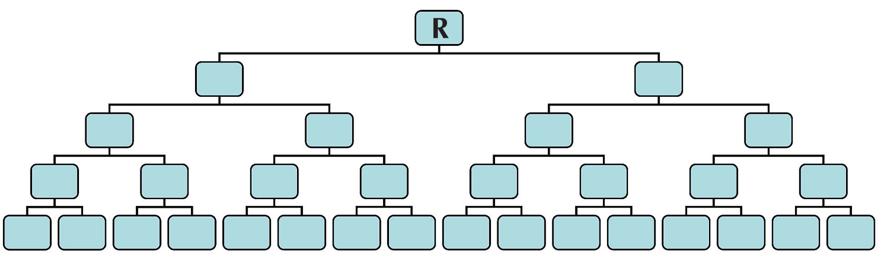
Complete the table with the initials of the foods.

Soup								
Main c.								
Dessert								

Maximum how many members can the Falánk family have if there were not two family members who ordered the same three courses?

In another case the tree diagram is of help:

Évi is threading pearls. She has yellow, red and blue pearls. She can put five pearls on a string. First she always threads the red pearl. Same colour pearls cannot be put next to each other. In how many different ways can Évi thread the pearls? Draw into the figure the initials of the colours, showing what options she has for making the row of pearls.



Detailed Assessment Frameworks of Grades 5-6

Numbers, Operations, Algebra

At the beginning of grade 5 the mathematical preparedness of the learners in this domain should be evaluated carefully. The most reliable data about the preparedness of the learners can be received if the assessment of their skill is preceded by repetitions through manifold, varied activities.

Numbers, number systems

By the end of grade six children learn the rational numbers. The extension of the number circle (wholes, fractions, decimal numbers), within this the interpretation of the negative numbers, two types of interpretation fractional numbers (for example, 2/3 may mean that we divide a whole cake into 3 equal parts and we take 2 pieces of these parts, or we take 1 third of two whole cakes – equal to the above), the learning of the definition of opposite, absolute value, examination of the properties of numbers (for example, parity, neighbouring numbers, divisional options, etc.) make the children capable to write down and read the learned numbers in the correct way, they understand and are able to use fractions, decimal numbers and negative numbers.

At these grades the learners are interpreting the concept of rounding in extended number circles and they use the rules of rounding. They learn the definition of percentage, base, interest rate. The discussion of experiences collected from other fields of literacy (for example, subjects of natural sciences) is also important because it serves the enrichment of these definitions. Similarly, the natural sciences build on the knowledge of mathematics, and through knowledge transfer the computation skill has a decisive role in the biological, chemical, physical and geographical calculations, too.

During the teaching of number systems (decimal base and binary system which is only demonstrated in grades 5-6 and is not a requirement in the curriculum) it can be made clear to what extent the placing (place value) of numbers (form value) influences the real value of the number. Here there is again an occasion for the introduction of the role of 0 as replacement of a place value. In these grades the reliable knowledge of decimal system is already a requirement.

Fill in the table based on the example.

Decimal system									Writing of numbers	Writing of numbers with words
...	10^3	10^2	10	1	1/10	1/100	1/1000	...		
	2		1			3			2010.03	Two thousand ten point three hundred
									207.8	
										Seven thousand eight hundred seven

During the studying of the properties and division possibilities of whole numbers we can learn the simpler rules of divisibility. *The knowing, and the application of divisibility in problems by 2, 5, 10, 4, 25, 100 is formulated as a requirement.* It is important to deal with parity, divisibility of 0, too. Tasks related to the number theory are able *to develop, improve the need for proving* (for example, How can we prove that divider of the odd numbers is odd?).

During the years *the many different representations, denotations of numbers also improves the combinatorial reasoning of learners* (for example, $6 = 3 + 3$ (can be divided into two equal parts, therefore the whole is even) $=$ 2×3 (there is 2 in the division to members, therefore the whole is even) $=$ $4 + 2 = 7 - 1 = 4 - (-2) =$ etc.).

The task below also proves that the representation of numbers in different forms plays a great role in the understanding of the subtracting of negative numbers.

Operations

We extend the arithmetic operations to the growing circle of numbers, the properties of which are inherited, the operation concept is deepening, and the operational algorithms are recognized. The root of the apperception of operations with wholes, fractions, and decimal numbers of different prefixes, be they made verbally or in writing, is the correct understanding of decimal system (place value, formal value, real value).

In connection with the operations mention should be made of the role of numbers 0 and 1 in the operations. For example the following examples show the consequences of the use of 0 as a multiplier factor:

Calculate the result of the following operations.
2 · 3 · 2 · 7 · 5 · 0 · 4 · 6 = ?
Solution: *The result of the multiplication is 0, since if one factor of the product is 0 the result is 0. Use fewer factors in case of paper-and-pencil tests.*

5	**13**	**9**	**8**	**7**
0	**7**	**4**	**11**	**22**
3	**32**	**0**	**6**	**18**
27	**2**	**4**	**0**	**9**
8	**12**	**19**	**5**	**3**

We have written whole numbers in the fields of the 5 × 5 grid shown on the figure.
Draw a thick continuous line along the grid lines so that it starts from the grid point of one of the border line of the quadratic grid and arrive at the grid point of another border line. The product of numbers on one side of the thick line be equal to the product of numbers on the other side.

Solution: *On both sides of the thick line there should be a 0 number. There are several solutions.*

During the years the role of 0 and 1 in the operations is gradually recognized, the knowledge of operational properties and their use become aware. The preliminary estimation, calculation and checking of results and the comparison of the results of computation, discussion of the possible causes of differences improves the computation skills, the algorithmic reasoning, the estimation ability and the need for self-checking. The correct keeping of the sequence of operations requires consistency, concentration.

The children should be able to multiply and to divide positive fractions by positive whole numbers, they should understand the basic operations and operational properties in the circle of rational numbers, and they should know and use the correct order of operations.

Put operational symbols and parentheses between the numbers so that you get the specified result.

3 7 3 3 = 4 Solution: 3 · (7 – 3) : 3 = 4
12 3 9 99 = 43 Solution: 12 · (3 + 9) – 99 = 44

Algebra

In the extended number circle besides problems containing numerals the equations and inequalities also appear in the open sentences (open sentences where the predicate is equal, smaller, bigger, smaller or equal, bigger or equal). The unknowns are in general marked by a letter. When making operations by letters we formulate conditions concerning the numbers to be written in the place of the letters (for example, in the case of 5/b the b cannot be 0). With the expressions received by using numbers, letters (unknowns) we formulate operations (for example, 3a; –2b; c/4), relations between operations and search for the solutions (solution set or truth set). These activities prepare the subject of algebra being an independent topic in the higher grades.

Solve the equation by trial and error method on the basic set consisting of numbers 1; 3; –2; 0; 5; –4.
$2a+(-4)=6$

Think of a number. Add 7 to it. Subtract the double of the result. Take the double of the result. Subtract 14 from it. From the received result subtract the originally thought number.
If you computed well you got the thought number as a result.
Why?
Solution:
We can confirm the validity of the above statement if we exactly follow the instructions in the language of mathematics.
Mark the thought number by x.

The sequence of instruction is: $(x + 7) \cdot 2 - 14 - x = x$,
and the equality is true.

Relations, Fractions

The proportionality tasks have practically been present since the beginning of schooling. The concept of fraction, the comprehension and practicing of multiplication, comparison, certain number theory questions, measurement, conversion of units, determination of perimeter, area are all based on proportional reasoning. Tasks connected to sequence often occur since the repeated execution of certain operations results in a row of numbers containing some kind of regularity. The recognition of regularities, their use serve from the beginning the improvement of the ability to follow the rules and to make inferences. In other subjects the natural, physical phenomena, the studying of timeline changes in processes, the mathematical description of the cause-effect relation, modeling of the everyday life give the basis to the development of the concept of functions.

As to direct proportionality there are several possibilities for the selection of the task. Every unit conversion, shopping, uniform motion, work, sale, interest rating, enlargement, scale of map, comparison of areas, etc. are suitable for the formulation of routine tasks. Examples:

A petrol tank of a car can receive 47,5 l petrol. We fill the petrol by a 2,5 l can. How many cans do we have to pour so that it be full?

We bought 8,5 kg of apples for 340 Ft. How much does 12 kg of this type cost? What is the relationship between the price of the apple and its weight?

Zoll (inch) is a German unit of length, 10 inches=254 mm. How many mm is the diagonal of the computer screen if it is 15 inch long?

On a map of scale 1 : 30 000 000 the distance between Budapest and London is 7 cm. In reality, how many km is the aerial distance between the two cities?

The population of a city has grown by 15% during one year. How many inhabitants lived in the city at the beginning of the year if the growth was 7500 persons?

The relations used in the case of direct proportionality in a different formulation are appropriate for the use and testing of the concept of inverse proportionality. The textbooks mainly contain tasks about work, distribution of costs-profits, relations between time and speed needed to making a given route, length of lateral face of a rectangle of specified area.

A family preserves raspberry juice for winter. If they fill the juice into half liter bottles, they need 21 bottles. How many would they need of 7 dl bottles?

An express train travelling at an average speed of 80 km/h takes the distance between two cities in one and a half hour. How long does the route last between the two cities by a local train if this train travels at an average speed of 45 km per hour?

4 people can finish a work in 12 days. With the same pace of work in how many days can 6 persons finish the work?

How long can the the sides of a rectangle with area of 24 cm^2 be, if its sides expressed in cetimeters are integers? Write in the table.

a (cm)											
b (cm)											

When studying the different effects and events the children can record the recognized, collected data in different ways (by text, formula, table, diagram, graph). The different solutions can be converted into each other. They are able to determine locations in practical situation, in specific cases. At this age mainly problems relating to motions, change of temperature, standing of water level are suitable for representing relations, connections on diagrams, graphs.

Mark on the number line, that:
- *it is warmer than –2 ºC*
- *it is not colder than –4 ºC, but the temperature is below freezing point.*

We are cooling water of 40 ºC. The temperature is decreasing by 6 ºC per minute. Make a table and a graph about the change of water temperature. Formulate in mathematical language how the temperature of water (T) depends on the time passed (t).

Geometry

In grades 1-4 we lay the foundations of geometrical definitions and knowledge. Here the procedures connected to actions, experiences are dominating. As a continuation of the work started in the lower grades the introduction, comprehension, ripening of concepts is accompanied by a lot of carefully planned activities in the upper grades.

Grades 5-8 are connecting the years of approach formulation, activity and discovery inducing works of lower grades and the work of grades 9-12 developing, teaching deductive reasoning. In the teaching of mathematics in the upper grades great emphasis should be placed both on the specific, practical activities in the integration of the children's experiences into teaching and on the development of abstract reasoning. Although the accents are gradually moved from the specific activities to abstraction, this dual approach is present parallel through the upper grades.

In the upper grades in teaching geometry and measurement topics it is always important that children could always return from the abstract concepts to the specific, practical meanings and certainly vice versa and that they could discover the general in the world of specific effects. Using the terms of the realistic mathematical movement: geometry is an excellent domain for the development of horizontal and vertical mathematical activities.

The geometrical topics of the lower grades get a role in the upper grades, too and we continue to use the methods of the lower grades. Diversified collection of experiences, use of tools, playfulness and games assist the building of concepts from the specific to the general. The children's reasoning is primarily inductive, but the need for generalization gradually comes into the foreground.

During teaching special attention should be paid to the development of the ability of regular trying, estimation, checking, pre-planning of solutions, and to the comprehensible description of the sequence of solution.

In the description of the geometrical frameworks of grades 5-6 we follow the partial content domains learned in the lower grades. In line with the traditions of the mathematical curriculum the children meet with the different definitions several times during their school years, thus in many cases seemingly the requirements of the lower grades are repeated. On the verbal level of the geometrical concepts this is the case, on the other hand the manipulative activities and visual memories playing a decisive role in the development of geometrical concepts make possible that the already acquired concept build of abstraction the mathematical knowledge of the learners further on a higher level. Of the four geometrical partial areas of the former grades orientation is not playing a role as a separate field in grades 5-6. The knowledge elements which could be listed in that category has no differentiating role in the assessment in this age group or can be put into the measurement category.

Constructions

A list of requirements: getting acquainted with spatial geometric elements, their relative positions, the concepts of parallelism, perpendicularity, the properties of rectangle (square), cuboids (cube), nets of cuboids, the pictorial view and properties of polygons, characteristics and categorization of triangles, rectangles.

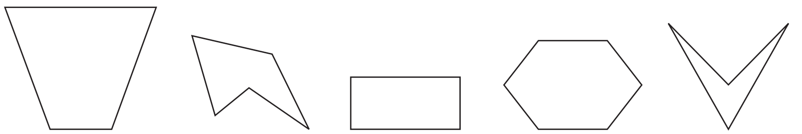
Children learn the concept and measurement of distance and angle. The looking for points with given specificities evokes the definitions of perpendicular bisector of the segment, circle and sphere, solution of construction tasks. Children are able to create plane shapes and bodies on manipulative and pictorial levels. Convexity appears among the known geometrical properties. Learners are able to group plane shapes and bodies on the basis of the learned geometrical properties.

They should know the properties and body nets of cube and cuboid and the basic properties of triangles and squares. The development of the concept of circle and sphere, and knowledge about basic properties is also a requirement.

As to the use of compasses and ruler it is required from the learners to be able to copy a segment, to draw parallel and perpendicular lines with two rulers, to copy angles and to construct median perpendicular to a segment.

The learners have to know the definition of angle, the different types of angles and learn how to use the angle-meter. In grades 5-6 learners are able to use precisely the terms point, straight and segment.

Group the plane shapes below according to their being convex or concave. Write the corresponding letters on the dot line.



a) *b)* *c)* *d)* *e)*

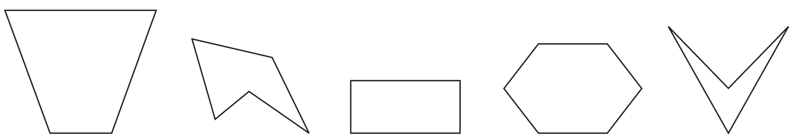
Convex polygons: ....

Concave polygons: ....

Transformations

Students will be able to construct the mirror-image of well-known shapes. They have to recognize shapes symmetrical around the axis. Symmetry should be recognized on specific examples taken from everyday life and from art. They are able to formulate by words the properties of projection on the axis.

Of the plane figures below which one has mirror axis? Circle the letter of the one which has minimum one mirror axis and cross the letter of the one without mirror axis.



a) *b)* *c)* *d)* *e)*

Measurement

In grades 5-6 the measurement index numbers can be used in the extended number circle. On the one hand this means that when making conversion of units it is required to convert not only the neighbouring units, but also the more distant ones, but thus we deal not only with measurements which are known or can be reconstructed from the everyday life, but many tasks of conversion of units turn into simple computation tasks. The extended number circle at the same time means that fraction numbers are included in the perimeter, area and volume calculations, as well as a new operation; the squaring is also applied in the geometrical computations.

At this age group children are able to compute the perimeter of triangles and squares, the surface and volume of cubes and of cuboids. Not the knowledge and the use of non-general formulas are required, but they should be able to make computation with specific, known or to be determined number data.

The children should know the standard units of measurement of length, area, mass, cubic content, volume and time. They have to make unit conversions in the number circle up to million. They should be aware of the volume and cubic content units and should convert them into each other.

They use the knowledge gained in grade 5 to the calculation of volumes and surface areas, they determine the surface and volume of bodies built of cuboids and cubes. In grade 6 they use knowledge acquired in the previous grades in area computation tasks which can be led back to the area of rectangle or get acquainted with the calculation of the area of right-angle triangle and mirror triangle, of convex and concave kite, rhombus, squared.

In connection with measurement activities we connect the area of measurements through preliminary estimations to the everyday experiences.

The simplest measurement tasks where the learned mathematical terms and symbols are checked typically belong to the following basic types:

Conversion of units

125 cm = ... mm
40 hl = ... cl
117 000 cm = ... km

Area and perimeter calculations

Calculate the perimeter of the rectangle the shorter lateral face of which is 2 cm, while the longer face is 3 cm.

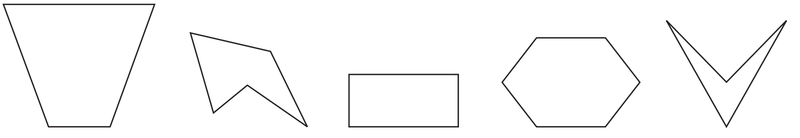
What is the length of the sides of the square with area 49 m^2?

Volume calculation

What is the volume of the cuboid the height of which is 6 cm, the other two edges are 8 and 10 cm?

In the case of simple word problems of measurements the wording of the problem determines the method of measurement, or the consecutive operations to be made with the received numbers.

With the help of your ruler measure the perimeter of the below shapes. Put them in descending order on the basis of their perimeter.



a) b) c) d) e)

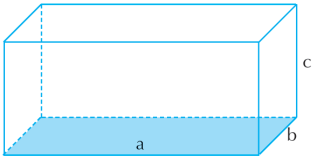
Letter mark of the shapes according to the size of their perimeter in descending order: ....

Perhaps in connection with the next task we have to explain why we regard it a simple, routine word problem and why we do not consider it a realistic problem. The key is that the numbers and geometrical terms contained in the problem lead to the solution without their comparison with the everyday experiences. It is not necessary to make a model with the help of mathematical symbols and concepts about the problem situation, but we mainly look for the mathematical definitions and operations contained in the text of the problem. Although the dimensions of the pool contained in the text of the problem can be compared with the everyday life, but it can be seen that the given dimensions can be varied optionally in the known number system and the task will not be easier for the majority of learners if we give the dimensions of a standard garden pool or swimming pool.

The recently built pool for children in the community center for water sports is 0,5 meter deep, 10 meter wide and 15 meter long. How much water is needed to fill up the pool?

It should also be considered to what extent the drawing prepared to the problem, or the drawing expected as part of the solution modify the difficulty of the task. If we give a draft drawing to the previous problem, where we assign data to the three edges of the cuboid, we will remain in the routine word problem category in the same way, as when the learned rules have to be applied in the framework of mathematical concepts and symbol system.

The recently built pool for children in the community center for water sports is 0,5 meter deep, 10 meter wide and 15 meter long. How much water is needed to fill up the pool?



Compared to the problems written only by symbols the simple word problems of unit conversion can test the comprehension of texts. It is possible that not the same learners can solve correctly the following to tasks:

Version 1:

32 dm^3 = ... liter

Version 2:

Determine the cubic measure of the vessel the volume of which is 32 dm^3.

The above problems highlighted the assessment problems of the unit conversion skills. Part of the unit conversion tasks can be solved on the basis of cognition based knowledge, the other part requires the use of computation skill. It is also possible that it depends on the character of the problem setting how the learners handle the conversion problem, as a computation problem, or as a problem intending to check his/her knowledge about units of measurements.

Combinatorics, Probability Calculation, Statistics

During the past decade there was significant change in the Hungarian mathematical education since the probability calculation and statistics topics were included in the maturation exam requirements, thus they had a reactive influence on the high school education and had significantly reshaped it. Nevertheless, it should be underlined that in the international educational surveys made since the 1960s, the descriptive statistics has been present from the beginning even in the assessment of the youngest age group, of children around 10 years of age. In the Hungarian education system the corresponding knowledge elements are related to the mathematical education, but also to the integrated subjects of education of sciences (environmental, natural studies).

Students – in the course of the solution of diversified problems and tests – learn terms which have the same use in mathematical and everyday contexts: case, event, and experiment.

They are able to determine and to represent on event tree or in table format the possible outcomes of the different probability tests. They use the terms of certain events and impossible events. The learner are aware of the terms of mutually exclusive and mutually non-exclusive events. They are able to represent the frequency of events in tables on different figures – frequency bar diagram, circle diagrams. They are able to determine events of the lowest and highest frequency and are able to calculate the arithmetic mean of some numbers. They are able to sort a disordered data set according to the frequency of occurred events or according to other criteria in the form of list, table and diagram.

At this age the probability concept is interpreted by the learners by the fraction „positive event / all events" and they can also formulate some remarkable probability values, too: the probability of an impossible event is 0%, the probability of a secure event is 100%, and the probability of two events with equal chances is 50-50%. They also meet with problems which can eliminate certain misconceptions. This can be for example the following: in a family with two children (supposing 50-50% probability of the birth of a boy or a girl) the probability of the case of one boy – one girl is not 1/3, but ½, or when we throw up a coin twice the probability of heads or tails is not 1/3, but 1/2.

Students know the dice as a visual tool of representation of unintentional events. They try by simple tests that if we throw by the dice many times, the six possible values on the dice will occur in approximately the same number.

Of the methods of computing all possible cases the children know in empirical way (without formal formula) the method of counting of options received with permutation without and with repetition in case of sets of less than ten elements; how to count the options gained by the repetition and without repetition variant if the end result remains in the hundred number system; the method of counting options received by the without repetition combination in the case of selection of partial set of the maximum six element set.

We present an example where the linguistic elements exclusively serve the mediation of the mathematical structure of the problem.

How many two-digit numbers can be made of numbers 1, 2 and 3 if we can use each number only once?

This task can be regarded a routine word problem if one of the numbers is changed to 0.

Parts of the requirements of descriptive statistics belong to the topics of fractions, relations, too. Analysis of graph, figures, reading of the most frequent value, scope of observable values are the required knowledge elements.

Combinatorics has traditionally been an integrated part of the Hungarian mathematical education. There have been refined traditions of setting tasks on manipulative, pictorial and symbolic levels where the number of options has to be determined. The major part of these textbook problems can be regarded routine tasks, since of the mathematical structures dressed in text form appear, but the everyday knowledge and experiences do not have real, relevant role. In such cases a typical problem setting strategy is the problem starting with „Anna, Béla, Cili and Dani…", where for example four different, equal activities can be associated to the children's names. Another typical solution is when topics alien to the children's experiences appear in the problem texts: water pipe systems, phone line networks, managerial appointments, etc.

Anna, Béla and Cili are siblings. One of them empties the garbage bin every day, the other waters the flowers. In how many different ways can they share the household chores?

As a consequence of the characteristics of this age group the teaching examples will be underrepresented compared to the realistic problems in the field of combinatorics. The solution without formula of the basic counting examples can be expected if the comprehension of the problem is assisted by memories or for example by drawing models.

About the Contributors

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Katalin Gábri
She graduated in mathematics, informatics and educational evaluation. She worked as a teacher for ten years. Later she became an advisor at the Pedagogical Institute of Nógrád County then in the Hungarian Gallup Institute. She conducted local and national surveys, school efficiency studies and research into the measurement of the added value in education. She regularly gives talks in in-service training on the evaluation of learners and institutes.

Józsefné Lajos
She is an expert in educational management, evaluation and accreditation. She graduated in mathematics and physics from Loránd Eötvös University. She has been involved in several mathematical development projects in

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She is a senior lecturer at the Department of Mathematics, Faculty of Primary and Preschool Teacher Training of Loránd Eötvös University. She graduated in mathematics, physics and descriptive geometry from Loránd Eötvös University. She had been working in public education for two decades. Her major research interest is the development of problem-solving in teaching geometry.

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Julianna Szendrei
Head of the Department of Mathematics, Faculty of Primary and Preschool Teacher Training of Loránd Eötvös University. She graduated in mathematics and physics and obtained a PhD degree in mathematical statistics. She was awarded Széchenyi professor fellowship between 2002 and 2004. She is a founding member of the mathematics didactics PhD school at the University of Debrecen, chairperson of the CIEAEM, the international organization of teaching mathematics for two periods. Her research topics are concerned with the kindergarten and young learners and also the mathematical thinking of their teachers.

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Senior lecturer in the Department of Mathematics, Faculty of Primary and Preschool Teacher Training of Loránd Eötvös University. She graduated from the Teacher Training College of Budapest and worked as a primary school teacher for eight years. She majored in mathematics at the Loránd Eötvös University. Her research interest is the probabilistic reasoning of preschool children and young learners. She has been involved in development of competency-based mathematics education. She is an active member of the Bolyai Society.

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